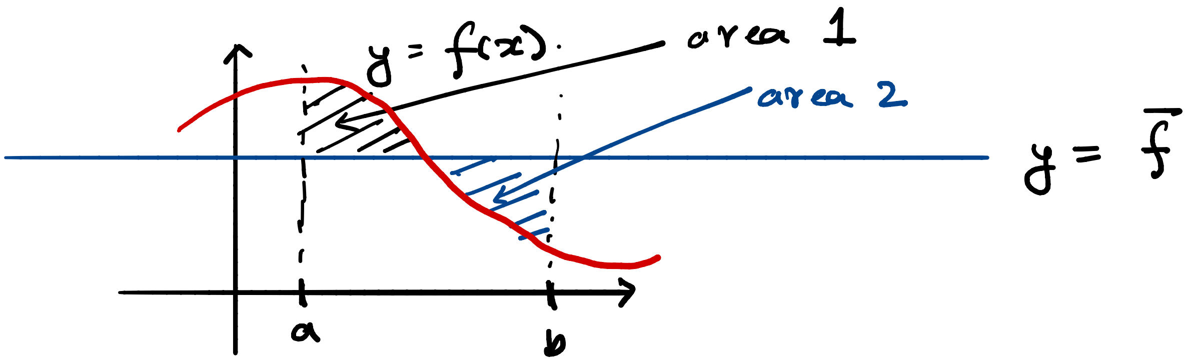
Average value. (CLP 2.2)

Goal: Compute average value of a function.



find $\bar{f} \in \mathbb{R}$ s.t. area 1 = area 2

$f(x_1), f(x_2), \ldots, f(x_N)$

Average.

Definition:

Consider data points $f_1, f_2, \dots, f_N$ _ (a collection of real numbers. The arithmetic mean f_{ave} is

$$f_{ave} = \bar{f} = \langle f \rangle = \frac{1}{N}(f_1 + f_2 + \dots + f_N) = \frac{1}{N}\sum_{i=1}^{N} f_i$$

discrete quantity: overge value of N samplings of a function.

Now suppose we want to define the "average" of a function $f(x)$ on $a \le x \le b$.

Riemann sum:

Average (contd.)

Find the average of $f(x)$ on $[a,b]$.

- Partition $[a,b]$ into N equal segments $[x_{i-1}, x_i]$.
  $\left(x_0 = a, \quad x_N = b, \quad \Delta x = \dfrac{b-a}{N} \right)$
- Pick $x_i^* \in [x_{i-1}, x_i]$.
- Compute mean of $f(x_1^*), f(x_2^*), \ldots, f(x_N^*)$:

$$f_{ave} = \bar{f} = \langle f \rangle = \frac{1}{N} \sum_{i=1}^{N} f(x_i^*).$$

- Multiply and divide by Δx:

$$f_{ave} = \frac{1}{N \cdot \Delta x} \sum_{i=1}^{N} f(x_i^*) \Delta x \approx \frac{1}{b-a} \int_a^b f(x)\,dx$$

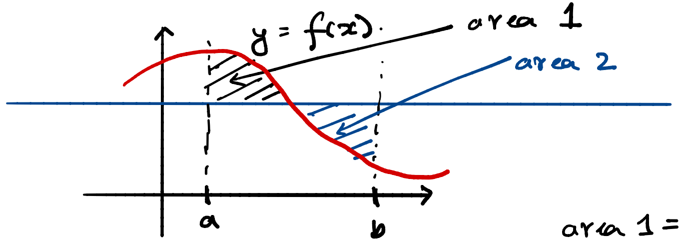
Mean of integrable function.

Definition.

Let f be an integrable function, then the average value $\bar{f}$ of $f(x)$ for x in $[a,b]$ is

$$f_{ave} = \bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

For positive (or negative) function:



$$\bar{f}(b-a) = \int_a^b f(x)\,dx$$

$$y = \bar{f} \iff \int_a^b \bar{f}\,dx = \int_a^b f(x)\,dx$$

area 1 = area 2.

Example 1

Find the average of $f(x) = x^2$ on $0 \le x \le 3$.

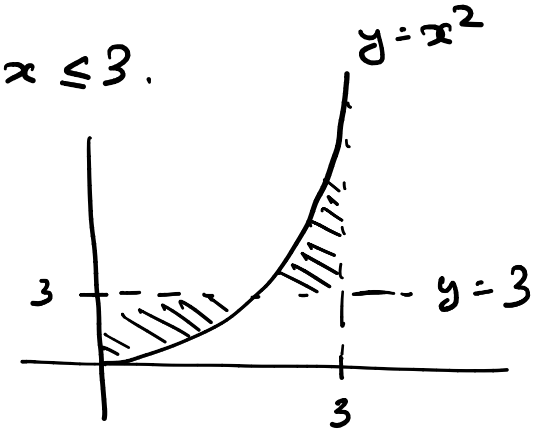
By definition:

$$\bar{f} = \frac{1}{3-0} \int_0^3 x^2 \, dx$$

$$= \frac{1}{3} \left. \frac{x^3}{3} \right|_0^3$$

$$= \frac{27}{9}$$

$$= 3.$$



shaded area are equal

Example 2.

Let $v(t)$ be the speed of a particle on a time interval $0 \le t \le T$. What is the average speed?

$$\bar{v} = \frac{1}{T}\int_0^T v(t)\,dt$$

$$= \frac{1}{T}\int_0^T \frac{dx}{dt}\,dt$$

$$= \frac{1}{T}\Big[\, x(t) \,\Big]_0^T$$

$$= \frac{x(T) - x(0)}{T}$$

average velocity is distance travelled over total time.

Example 3.

A cup of tea has a temperature of 95°C in a room where the temperature is 20°C. Newton's law of cooling states the temperature of tea after time t obeys

$$T(t) = 20 + 75e^{-t/50}$$

What is the average value during the first half hour?

Soln:

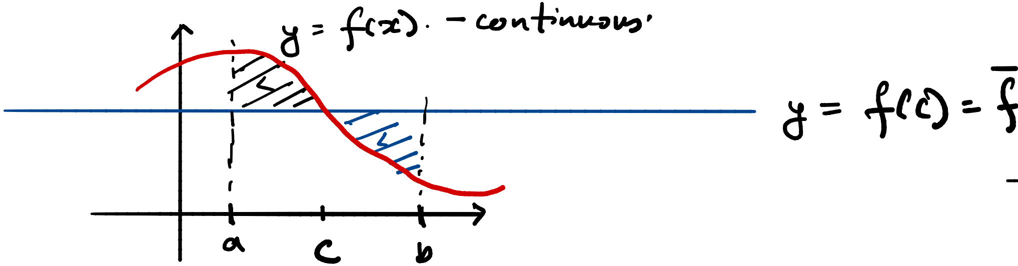
$$T_{ave} = \frac{1}{30}\int_0^{30}\left(20 + 75e^{-t/50}\right)dt.$$

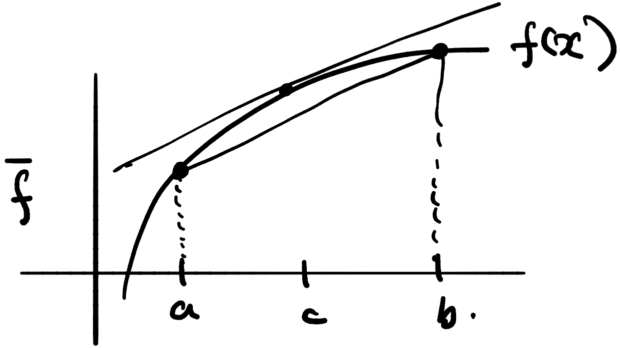
$$= \frac{1}{30}\left[20x - 75.50\, e^{-t/50}\right]_0^{30}$$

$$= 20 - \frac{75.50}{30}e^{-3/5} - \left(0 - \frac{75.50}{30}\right)$$

$$= 76.4^\circ C.$$

Mean value theorem for integrals.





<u>Theorem</u>: If $f(x)$ is continuous on $[a,b]$ then there exists a number $c \in [a,b]$ s.t.

$$\int_a^b f(x)\,dx = f(c)\cdot(b-a)$$

Interpretation: $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx = f(c)$.

somewhere on $[a,b]$, the function takes average value.

Example:

Find the number c so that the MVT for integrals is satisfied for $f(x) = x^2+3x+2$ on $1 \le x \le 4$

average value $= \dfrac{1}{3}\displaystyle\int_1^4 (x^2+3x+2)\,dx$

$$= \frac{1}{3}\left[\frac{x^3}{3} + \frac{3x^2}{2} + 2x\right]_1^4$$

$$= \frac{1}{3}\left[\frac{64}{3} + \frac{48}{2} + 8 - \frac{1}{3} - \frac{3}{2} + 2\right]$$

$$= \frac{99}{2}.$$

$$c^2 + 3c + 2 = \frac{99}{2} \Rightarrow c^2 + 3c - \frac{95}{2} = 0$$

$$c = \frac{-3 \pm \sqrt{67}}{2} \quad \left(= \frac{-3 \pm \sqrt{9^2 - 4\cdot(-\frac{95}{2})}}{2\cdot 1}\right)$$