Series.

Consider a sequence $\{a_n\}$. Let's add up the terms to get

$$a_1 + a_2 + a_3 + \dots$$

This is called an infinite series and is denoted by $\sum_{i=1}^{\infty} a_n$.

We want to see whether the infinite series $\sum_{i=1}^{\infty} a_n$ is finite (converges) or is infinite (diverges).

Example.

Consider the infinite series $1+2+3+\dots = \sum_{i=1}^{\infty} i$

Lets look at the partial sum of $1+2+3+\dots$

$s_1 = 1, \quad s_2 = 1+2 = 3, \quad s_3 = 1+2+3 = 6, \dots$

$$s_n = 1+2+\dots+n = \frac{n(n+1)}{2}$$

The sequence given by $\{s_n\} \to \infty$ as $n \to \infty$. So, they diverge

Example

$\left\{\frac{1}{2^n}\right\}$

Consider the infinite series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}+\cdots$

Again, let's look at the partial sum:

$s_1 = \frac{1}{2}$, $\quad s_2 = \frac{1}{2}+\frac{1}{4} = \frac{3}{4}$, $\quad s_3 = \frac{1}{2}+\frac{1}{4}+\frac{1}{8} = \frac{7}{8}$, $\ldots$

$$s_n = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n} = \frac{2^n-1}{2^n}$$

Since $\frac{2^n-1}{2^n} = 1-2^{-n} \to 1$ as $n \to \infty$, we get

$\sum_{i=1}^{\infty} \frac{1}{2^n}$ converges to 1.

Partial sum.

$\left\{ \frac{1}{n} \right\}$

Now that we have looked at two examples, let's think about the "harmonic" series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$

Definition. Let $a_1 + a_2 + a_3 + \ldots$ be an infinite series and let

$$s_n = a_1 + a_2 + a_3 + \ldots + a_n$$

denotes its n^{th} partial sum. If $s_n \to s$ for some finite real number s, we write

$$a_1 + a_2 + a_3 + \ldots = s \quad \text{or} \quad \sum_{i=1}^{\infty} a_i = s.$$

and say that the series converges. If $\{s_n\}$ does not converges then we say $\sum_{i=1}^{n} a_i$ diverges.

Arthemetic operations:

Theorem (CLP Thm 3.2.9). Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences with $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$. Further let c be any real number. Then

$$\sum_{n=1}^{\infty} (a_n + b_n) = A + B. \qquad \{a_n + b_n\}$$

$$\sum_{n=1}^{\infty} c\, a_n = c A.$$

$$\sum_{n=1}^{\infty} (a_n - b_n) = A - B.$$

Remark: product and ratios are not so simple to calculate.

Geometric series.

One of the simplest type of infinite series that we can analyze are geometric series, which have the general form

$$\sum_{n=1}^{\infty} a r^{n-1} = a + ar + ar^2 + ar^3 + \dots \qquad \sum_{n=1}^{\infty} r^{n-1}$$

The main result for geometric series is as follows:

Theorem: Consider the geometric series $\sum_{n=1}^{\infty} a r^{n}$. If $|r| < 1$, then it converges and

$$\sum_{n=1}^{\infty} a r^{n-1} = \frac{a}{1-r}.$$

If $|r| \geq 1$ and $a \neq 0$ then the series diverges.

Geometric series (proof).

proof of theorem: We start with the n^{th} partial sum:

$$S_n = a + ar + ar^2 + \dots + ar^n \quad \text{--- (I)}$$

Now,

$$rS_n = ar + ar^2 + ar^3 + \dots + ar^{n+1} \quad \text{--- (II)}$$

Consider

$$S_n - rS_n = (a + \cancel{ar} + \dots + \cancel{ar^n}) - (\cancel{ar} + \cancel{ar^2} + \dots + \cancel{ar^n} + ar^{n+1})$$

so,

$$S_n = a(1 - r^{n+1})/(1-r)$$

Since $|r| < 1$ implies $r^n \to 0$ and so the result follows.

$$S = \frac{a}{1-r}$$

If $|r| > 1$, clearly $r^n \to \infty$ and it diverges.

If $r = 1$, $S_n = \underbrace{a + a + a \dots + a}_{n\text{-times}} = na \to \infty$

If $r = -1$, $S_n = a - a + a - \dots$ which oscillates.

Example.

Lets analyze the infinite series $9 - \frac{27}{5} + \frac{81}{25} - \frac{243}{125} + \dots$

Does it converge? First realize that its a geometric series. the parameters are:

ar

$$a = 9, \quad r = \frac{3}{5} \text{ or } -\frac{3}{5}$$

Since the sign alternates, $r = -\frac{3}{5}$.

Since $|r| < 1$, it converges and the sum equals:

$$\frac{a}{1-r} = \sum_{i=1}^{n} 9 \cdot \left(\frac{-3}{5}\right)^{n-1} = \frac{9}{1-\left(-\frac{3}{5}\right)} = \frac{45}{8}$$

Example

Calculate $\displaystyle\sum_{n=1}^{\infty} \frac{4^n + 5^n}{9^n}$ $\qquad \displaystyle\sum_{n=1}^{\infty} ar^{n-1}$ $\qquad \displaystyle\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$

Sol^n Let's use property of infinite series to simplify:

$$\sum_{n=1}^{\infty} \frac{4^n + 5^n}{9^n} = \sum_{n=1}^{\infty} \frac{4^n}{9^n} + \sum_{n=1}^{\infty} \frac{5^n}{9^n}.$$

$$= \sum_{n=1}^{\infty} \frac{4}{9}\left(\frac{4}{9}\right)^{n-1} + \sum_{n=1}^{\infty} \frac{5}{9}\left(\frac{5}{9}\right)^{n-1}$$

$$= \frac{4/9}{1 - 4/9} + \frac{5/9}{1 - 5/9}$$

$$= \frac{4}{5} + \frac{5}{4}$$

$$= \frac{41}{40}.$$

Example.

Calculate: $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{8^{2n+1}}$ $\qquad \displaystyle\sum_{n=1}^{\infty} a r^{n-1}$

Again, lets use $\displaystyle\sum_{n=0}^{\infty} c\, a_n = c \sum_{n=0}^{\infty} a_n.$

So, $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{8^{2n+1}} = \sum_{n=0}^{\infty} \frac{1}{8} \cdot \frac{3^n}{8^{2n}} = \frac{1}{8} \sum_{n=0}^{\infty} \left(\frac{3}{8^2}\right)^n$

So, $\displaystyle\sum_{n=0}^{\infty} \left(\frac{3}{64}\right)^n = \frac{1}{1 - 3/64} = \frac{64}{61}$

and putting it all together: $\displaystyle\frac{1}{8} \sum_{n=0}^{\infty} \left(\frac{3}{8^2}\right)^n = \frac{1}{8} \cdot \frac{64}{61} = \frac{8}{61}$

Telescoping infinite series:

A second type of infinite series that is easy to analyze is a telescoping infinite series, which has the general form

$$S = \sum_{n=1}^{\infty} (a_n - a_{n+1}) = (a_1 - \cancel{a_2}) + (\cancel{a_2} - \cancel{a_3}) + (\cancel{a_3} - \cancel{a_4}) + \dots$$

Thm: Suppose $S = \sum_{n=1}^{\infty} (a_n - a_{n+1})$ with $\lim_{n\to\infty} a_n = L$.

Then $S = a_1 - L$ and so S is convergent.

Example.

Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges.

If it converges find the sum.

Soln: Let's write out a few of the partial sum:

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$$

$$S_3 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{9}{12} = \frac{3}{4}$$

$$\vdots$$

$$S_n = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \dots + \frac{1}{n(n+1)} = \frac{n-1}{n} . \to 1$$

$$\left\{ \frac{1}{n(n+1)} \right\} = \left\{ \frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \dots \right\}$$

since $\frac{1}{n} \to 0$.

Example (Contd.)

Again, lets look at $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

using partial fractions: $\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} = \frac{1}{n} - \frac{1}{n+1}$

so, $S_n = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \dots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$

$= 1 - \frac{1}{n+1}$

so, $S_n = \sum a_n - a_{n+1} = a_1 - L.$

$= 1$

Example

Discuss convergence of $\sum_{n=1}^{\infty} \log\left(1+\frac{1}{n}\right)$.

we can write the infinite series as:

$$\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right) = \sum_{n=1}^{\infty}\left(\log(n+1) - \log(n)\right)$$

which looks like telescoping sum but we must be careful:

$$S_n = \log(n+1) - \cancel{\log(n)} + \cancel{\log(n)} - \cancel{\log(n-1)} + \dots - \log(1).$$

$$= \log(n+1) - \cancel{\log(1)}.$$

So the series diverges as $\log(n+1) \to \infty$ as $n \to \infty$.

In our example, $a_n = \log(n)$ which does not vanish.