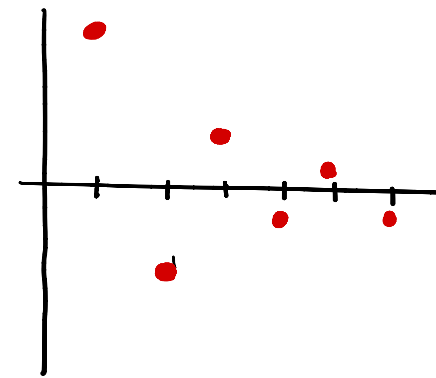
Alternating series.

Last time we looked at integral test.

Recall, integral test only works for series $\sum_{n=1}^{\infty} a_n$

with $a_n > 0$ and a_n decreasing.



What about series like

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} \quad \text{or} \quad \sum_{n=1}^{\infty} (-1)^{n-1} b_n, \; b_n > 0$$

that alternate between + and − ?

Alternating series test:

Theorem (CLP 3.3.14) Consider the alternating series:

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \dots \quad \text{with } b_n > 0 \ \forall n.$$

If

- $b_n \geq b_{n+1}$ for all $n \geq N_0$, for some N_0 (i.e tail is decreasing)
- $b_n \to 0$

then the series $\sum_{n=1}^{\infty} (-1)^n b_n$ converges.

Alternating series test

Remarks

i. A proof is given in CLP 3.3.10.

ii. We only need $b_{n+1} < b_n$ for large enough n.

because
$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = \underbrace{\sum_{n=1}^{N_0-1}(-1)^{n-1}b_n}_{\text{finite.}} + \sum_{n=N_0}^{\infty}(-1)^{n-1}b_n.$$

iii. A good way to check if b_n is decreasing is by finding the function $f(x)$ s.t. $f(n)=b_n$. Then look at derivative of $f(x)$.

Examples 1

1. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converge?

Remark: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverge by p-test.

Sol^n: Let $b_n = \frac{1}{n}$. Note that $b_n \to 0$., $b_n > 0$

also, $b_{n+1} \leq b_n$ because $\frac{1}{n+1} \leq \frac{1}{n}$.

by alternating series test, $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ converges.

There is enough cancellation for convergence.

Example 2

Does $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\log n}{n^2}$ converge? $\quad \frac{\log x}{x^2}$.

Sol^n: let $b_n = \frac{\log n}{n^2}$ $\therefore b_n \geq 0$ for $n \geq 1$. $b_n \to 0$.

define $f(x) = \frac{\log x}{x^2}$. note that $f(n) = b_n$.

Now $f'(x) = -2x^{-3} \log x + x^{-3} = x^{-3}(1 - 2\log(x))$

want $f'(x) < 0 \Rightarrow 1 - 2\log x < 0 \Rightarrow \log(x) > 1/2$

$\Rightarrow x > e^{1/2}$

take $x \geq 2$ then $f'(x) < 0$. So take $N_0 = 2$.

Then by alternating series test $\sum_{n=N_0}^{\infty} (-1)^{n-1} \frac{\log n}{n^2}$ converges.

Example 3

Does $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n}}{n+4}$ converge?

Sol^n: $b_n = \frac{\sqrt{n}}{n+4}$, $b_n \to 0$ and $b_n > 0$ for all n.

$f(x) = \frac{\sqrt{x}}{x+4}$, $f'(x) = \frac{1}{2\sqrt{x}} (4-x)(x+4)^{-2}$

need $4-x < 0 \Rightarrow x > 4. \Rightarrow$ pick $N_0 = 4$.

so, $b_n \geq b_{n+1}$ for $n \geq N_0$.

By alternating series test $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n}}{n+4}$ is

convergent.

Example 4.

Does $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^2+5}$ converge or diverge?

Sol^n Let $b_n = \frac{n^2}{n^2+5}$. $b_n \to 1 \neq 0$.

So, we cannot use alternating series test.

However, since $(-1)^{n-1} \frac{n^2}{n^2+5} \approx (-1)^{n-1}$ for large n which is not equal to 0. By basic divergence test, $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{n^2+5}$ diverges.

Example 5

Does $\sum_{n=1}^{\infty} (-1)^{n-1} \cos\left(\frac{\pi}{n}\right)$ converge or diverge?

$b_n = \cos\left(\frac{\pi}{n}\right)$, $b_n \to 1$ because $\frac{\pi}{n} \to 0$

and $\cos(x)$ is continuous at $x=0$.

So, we cannot use alternating series test. But from divergence test, $\sum_{n=1}^{\infty} (-1)^{n-1} \cos\left(\frac{\pi}{n}\right)$ is a divergent series.

Log 2

Show that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \log 2.$

<u>proof</u>: We know that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges by alternating series test.

Lets start with $N-1^{th}$ partial sum of geometric series.

$$S_{N-1} = 1 + r + r^2 + \dots + r^{N-1}$$
$$= \frac{1 - r^N}{1 - r}$$

$N \to \infty$

$S_N \to \frac{1}{1-r}$

assum $|r|<1$

Compare S_{N-1} and

$$r S_{N-1} = r + r^2 + \dots + r^N$$

Log 2 (cont'd).

Integrate both sides of $S_{N-1} = \dfrac{1-r^N}{1-r}$ on $-1 \le r \le 0$

$$\Rightarrow \int_{-1}^{0} (1 + r + r^2 + \dots + r^{N-1})\, dr = \int_{-1}^{0} \frac{1-r^N}{1-r}\, dr.$$

$$\Rightarrow \left[r + \frac{r^2}{2} + \frac{r^3}{3} + \dots + \frac{r^N}{N} \right]_{-1}^{0} = \int_{-1}^{0} \frac{1}{1-r}\, dr - \int_{-1}^{0} \frac{r^N}{1-r}\, dr$$

$$\Rightarrow 0 - \left((-1) + \frac{(-1)^2}{2} + \frac{(-1)^3}{3} + \dots + \frac{(-1)^N}{N} \right) = -\log(1-r)\Big|_{-1}^{0} - \int_{-1}^{0} \frac{r^N}{1-r}\, dr$$

$$\Rightarrow \sum_{n=1}^{N} (-1)^{n-1} \frac{1}{n} = \log 2 - \int_{-1}^{0} \frac{r^N}{1-r}\, dr.$$

goal : show $\displaystyle\int_{-1}^{0} \frac{r^N}{1-r}\, dr \to 0$ as $N \to \infty$.

Log 2 (contd)

Let $E_N = \int_{-1}^{0} \frac{r^N}{1-r}\, dr$ $\qquad u = -r, \quad du = -dr$

$$E_N = -\int_{1}^{0} \frac{(-u)^N}{1+u}\, du$$

$$= \int_{0}^{1} (-1)^N \frac{u^N}{1+u}\, du.$$

$$= (-1)^N \int_{0}^{1} \frac{u^N}{1+u}\, du.$$

it is sufficient to show $|E_N| \to 0$ to conclude

$E_N \to 0$. (why?) $\to$ squeeze theorem.

log 2 (contd).

Note that $1+x \geq 1$ for $x \in [0,1]$.

$$0 \leq |E| = \int_0^1 \frac{x^N}{1+x}\,dx \leq \int_0^1 x^N\,dx = \frac{1}{N+1} x^{N+1}\Big|_0^1$$

$$= \frac{1}{N+1}$$

as $n \to \infty$ $\frac{1}{N+1} \to 0$ so, $|E| \to 0$ (squeeze thm).

so, $E \to 0$ $\Rightarrow$ $\int_{-1}^{0} \frac{r^N}{1-r}\,dr \to 0$ as $N \to \infty$.

so, $\sum_{n=1}^{N} (-1)^{n-1} \frac{1}{n} = \log 2 - \int_{-1}^{0} \frac{r^N}{1-r}\,dr.$

$$\Rightarrow \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = \log(2)$$

Estimating remainder

Theorem: If $S = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$ is a convergent alternating series with $b_n \geq 0$, $b_{n+1} \leq b_n$, and $\lim_{n\to\infty} b_n = 0$, then the remainder R_N defined by

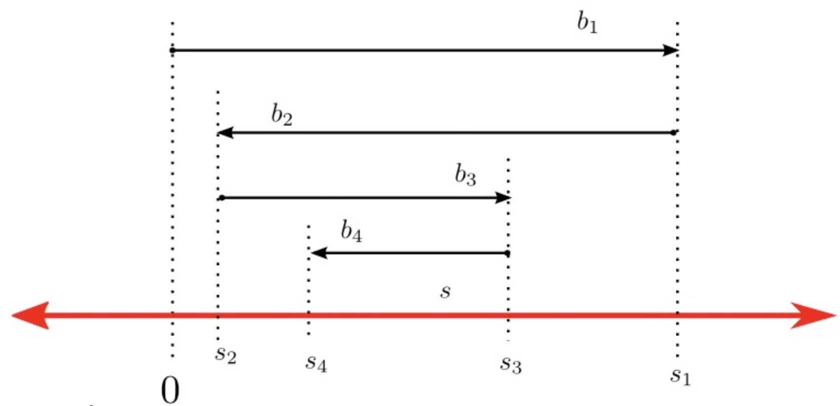
$$R_N = | S - S_N | \quad \text{with} \quad S_N = \sum_{n=1}^{N} (-1)^{n-1} b_n$$

satisfies bound

$$R_N \leq b_{N+1}$$

i.e. Remainder is bounded by first term after the N^{th} partial sum.

N^{th} partial sum of alternating series.



Consider $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} b_n$. with $b_n \to 0$

$b_n > 0$

$b_n > b_{n+1}$

Example 6

Recall from Taylor approximation. (n^{th} order approx.)

$$f(x) \approx \sum_{n=0}^{N} f^{(k)}(a)\,(x-a)^k \cdot \frac{1}{k!}$$

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(a)\,(x-a)^k. \longrightarrow \text{taylor expansion.}$$

$$f^{(k)}(x) = e^x$$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

How many terms in the infinite sum are needed to get an error of 2.5×10^{-8} for e^{-1}?

$$e^{-1} = \frac{1}{0!}(-1)^0 + \frac{1}{1!}(-1)^1 + \frac{1}{2!}(-1)^2 + \cdots$$

how many terms need

Example 6 (cont d.)

Sol^n Let $x = -1$, $e^{-1} = \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^n$. $b_n = \frac{1}{n!}$

Notice that $b_n \geq 0$, $b_n \geq b_{n+1}$ bcoz $\frac{1}{n!} \geq \frac{1}{(n+1)!}$

and $b_n \to 0$.

so, we need to find N s.t. $R_N = |S - S_N| < 2.5 \times 10^{-8}$.

$$R_N \leq \frac{1}{b_{n+1}} = \frac{1}{(N+1)!}$$

for $n = 10$, $\frac{1}{(n+1)!} \approx 2.5 \times 10^{-8}$. So, we need need 10 terms in the series to approximate e^{-1} to 2.5×10^{-8}.

Example

How many terms in the infinite series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is needed to approximate $\log 2$ with 10^{-5} accuracy?

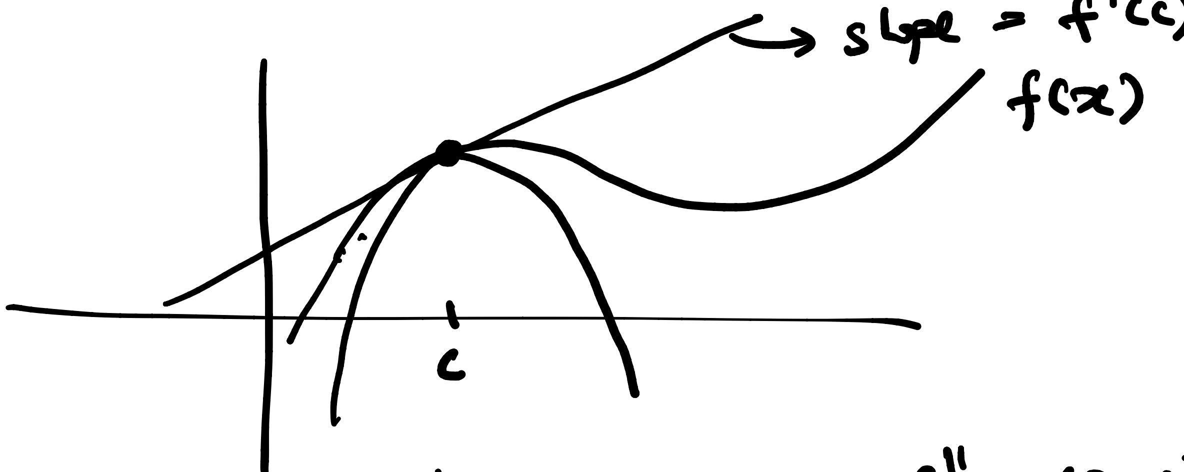
Let $b_n = \frac{1}{n}$. Note that $b_n > 0$, $b_n \to 0$, $b_n > b_{n+1}$.

The Remainder R_N satisfy $R_N < \frac{1}{N+1} := \frac{1}{b_{N+1}}$

So, to get an accuracy of 10^{-5} we need

$$\frac{1}{N+1} = 10^{-5} \quad \text{or} \quad N \approx 10^5$$

An enormous number of term since the series converges very slowly.



$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \cdots$$