Absolute and conditional convergence:

Recall that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges by alternating series test but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by integral test. $b_n = \frac{1}{n}$

Also, observe that $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge.

Definition Consider the series $\sum_{n=1}^{\infty} a_n$.

I. If $\sum_{n=1}^{\infty} |a_n|$ converges then we say the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent

II. If the series $\sum_{n=1}^{\infty} |a_n|$ is divergent but $\sum_{n=1}^{\infty} a_n$ is convergent, we say $\sum_{n=1}^{\infty} a_n$ is conditionally convergent

Comparision test:

A key property we established in comparision test was:

> **Theorem:** If $\sum_{n=1}^{\infty} |a_n|$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is also convergent. $a_n \leq |a_n|$

proof idea: let $S_N = \sum_{n=1}^{N} a_n$ and $T_N = \sum_{n=1}^{N} |a_n|$.

1. $T_N - S_N$ is bounded bone by $2T_N < \infty$
2. $T_N - S_N \geq 0 \quad \forall N. \Rightarrow$ monotonic.

so, By Monotonic Convergene theorem $T_N - S_N$ is convergent

$$\lim_{N \to \infty} T_N < \infty$$

Ratio test:

A key test for absolute convergence of a series $\{a_n\}$ $\sum_{n=1}^{\infty} a_n$ is the ratio test.

Thm: Let $N > 0$ be an integer and assume $a_n \neq 0$ for all $n \geq N$. Then

① If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, we have $\sum_{n=1}^{\infty} |a_n|$ converges. (and so does $\sum_{n=1}^{\infty} a_n$)

② If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ or $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$ then $\sum_{n=1}^{\infty} a_n$ diverges. ($\sum_{n=1}^{\infty} |a_n|$ also)

Ratio test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Remark:

a. Ratio test is useful when the series has exponent n, factorials, etc... like

$$\sum_{n=1}^{\infty} \frac{n\,2^n}{5^n}, \quad \sum_{n=1}^{\infty} \frac{e^n}{n!}, \quad \sum_{n=1}^{\infty} \frac{n^2 e^n}{4^n}, \text{ etc.}$$

b. It turns out ratio test is not as useful for series of the form $\sum_{n=1}^{\infty} \frac{P(n)}{Q(n)}$, $P(n)$ & $Q(n)$ are polynomial in n.

c. In ①, If $L<1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

$$\sum_{n=1}^{\infty} |a_n|$$

Remark (contd)

d. In ②, the series is not $\sum_{n=1}^{\infty} |a_n|$ absolutely convergent.

But $\sum_{n=1}^{\infty} a_n$ could be convergent, i.e $\sum_{n=1}^{\infty} a_n$ could be conditionally convergent

e. Important There is no conclusion if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$ (i.e. $L=1$). Then $\sum_{n=1}^{\infty} |a_n|$ may or may not be convergent.

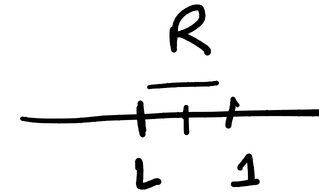
Further test is required.

Proof outline of ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$$

We first prove ①. Since $L<1$, we can pick a $R \in \mathbb{R}$ that satisfy $0 < L < R < 1$. Then there exists M so that for all $n \geq M$ we have $\left|\frac{a_{n+1}}{a_n}\right| < R$ because

$\left|\frac{a_{n+1}}{a_n}\right| \to L$ as $n \to \infty$.



Thus, $|a_{n+1}| < R|a_n|$ for all $n \geq M$.

So,

$$|a_{M+1}| < R|a_M|$$

$$|a_{M+2}| < R|a_{M+1}| < R^2 |a_M|$$

$$\vdots$$

$$|a_{M+p}| < R^p |a_M|$$

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad \text{if } |r|<1.$$

Then $$\sum_{p=1}^{\infty} |a_{M+p}| < \sum_{p=1}^{\infty} R^p |a_M| = \sum_{p=1}^{\infty} |a_M| R\, R^{p-1} = \frac{|a_M| R}{1-R} < \infty$$

proof (contd)

tail converges.

So, $\sum_{p=1}^{\infty} |a_{M+p}| < \infty$ and $\sum_{n=M+1}^{\infty} |a_n|$ converges.

So, $\sum_{n=1}^{\infty} |a_n| = \underbrace{\sum_{n=1}^{M} |a_n|}_{\text{finite}} + \sum_{n=M+1}^{\infty} |a_n| < \infty$. This proves ①.

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$

Now to prove ②, pick R in $1 < R < L$. Then for $n \geq M$, we must have

$$\left|\frac{a_{n+1}}{a_n}\right| \geq R \rightarrow |a_{n+1}| \geq R|a_n| . \quad \forall \ n \geq M.$$

Thus $|a_{n+1}| \geq |a_M|$ for $n \geq M$ since $|a_{n+1}| \not\to 0$ by basic divergence test. So, $\sum_{p=M+1}^{\infty} |a_{n+p}|$ is divergent.

$\boxed{\sum |a_n| \text{ is divergent}}$

$\sum_{n=1}^{\infty} a_n$ will diverge if $a_n \not\to 0$.

Example 1

Consider $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ where $a_n = (-1)^n \frac{1}{n}$.

By alternating series test $\sum_{n=1}^{\infty} a_n$ converges.

But we have $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges. So,

by our definition we conclude that $\sum_{n=1}^{\infty} a_n$ is

conditionally convergent.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{1}{n+1} \cdot n = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

So, we can't use ratio test to conclude.

Example 2

Consider $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ where $a_n = (-1)^n \frac{1}{n^2}$.

By alternating series test, we know that $\sum_{n=1}^{\infty} a_n$ converges.

By integral test, we know that $\sum_{n=1}^{\infty} |a_n|$ converges.

So, by definition we say that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ is absolutely convergent.

Example

Discuss absolute / conditional convergence or divergence of each of the following:

(a) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$

(b) $\sum_{n=1}^{\infty} (-1)^n \left(\frac{2n+1}{6n+5}\right)$

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n (n^2+1)}{n^4+5}$.

(d) $\sum_{n=1}^{\infty} (-1)^n n e^{-n^2}$

(e) $\sum_{n=1}^{\infty} \frac{n \sin(n)}{n^3+1}$

Example

ⓐ $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$.

$\lim_{n\to\infty} \frac{a_n}{b_n}$, $a_n = \frac{n}{n^2+1}$, $b_n = \frac{1}{n}$

Sol^n ⓐ First check $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n}{n^2+1}$. By comparision limit test (compare to $\frac{1}{n}$), we have that $\sum_{n=1}^{\infty} |a_n|$ is divergent. So, $\sum_{n=1}^{\infty} |a_n|$ is not absolutely convergent.

Now, check $\sum_{n=1}^{\infty} (-1)^n b_n$, where $b_n = \frac{n}{n^2+1}$.

Notice that b_n is decreasing for large n and $b_n > 0$ with $b_n \to 0$. Thus, by alternating test $\sum_{n=1}^{\infty} (-1)^n b_n$ is convergent.

Hence, $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$ is conditionally convergent.

Example

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{2n+1}{6n+5}$.

$$|a_n| = \frac{2n+1}{6n+5}$$

Sol^n Let $a_n = (-1)^n \dfrac{2n+1}{6n+5}$. Notice that

$a_n \approx (-1)^n \dfrac{1}{3}$ for large n. So, $a_n \not\to 0$

as $n \to \infty$. Thus, $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

Example

$\sum (-1)^n \frac{1}{n}$ $\qquad$ $\lim_{n\to\infty} \frac{a_n}{b_n}$

ⓒ $\sum_{n=1}^{\infty} (-1)^n \left(\frac{n^2+1}{n^4+5}\right)$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$

Sol^n: First check $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n^2+1}{n^4+5}$.

Since $\frac{n^2+1}{n^4+5} \approx \frac{1}{n^2}$ for large n, we expect convergence.

Let $b_n = \frac{1}{n^2}$. Notice that $b_n > 0 \ \forall \ n$.

and $\lim_{n\to\infty} \frac{n^2+1}{n^4+5} \cdot n^2 = \lim_{n\to\infty} \frac{n^4+n^2}{n^4+5} = 1$

Since limit exists and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, $\sum_{n=1}^{\infty} \frac{n^2+1}{n^4+5}$

is convergent by limit comparision test.

So, $\sum_{n=1}^{\infty} (-1)^n \frac{n^2+1}{n^4+5}$ is absolutely convergent.

Example

(d) $\sum_{n=1}^{\infty} (-1)^n n e^{-n^2}$ with $a_n = (-1) n e^{-n^2}$. $\sum_{n=1}^{\infty} n e^{-n^2}$

Sol^n: First test $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} n e^{-n^2}$. $\int_1^{\infty} x e^{-x^2} dx$.

By ratio test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1) e^{-(n+1)^2}}{n e^{-n^2}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1) e^{-n^2-2n-1}}{n e^{-n^2}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n+1}{n} e^{-2n-1} \right|$$

$$= 1 \cdot 0 = 0.$$

So, by ratio test, we conclude that $\sum_{n=1}^{\infty} (-1)^n n e^{-n^2}$ is absolutely convergent

Example

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{n\sin(n)}{n^3+1}$.

$$\sum_{n=1}^{\infty} \left| \frac{n\sin(n)}{n^3+1} \right|$$

Note that we expect $\displaystyle\sum_{n=1}^{\infty} \frac{n\sin(n)}{n^3+1}$ to converge because

$$\frac{n\sin(n)}{n^3+1} \approx \frac{1}{n^2} \quad \text{for large } n.$$

Note that we can't use integral test because $\dfrac{n\sin(n)}{n^3+1}$ is not always decreasing.

Also, we can't use limit comparision test because

$$\lim_{n\to\infty} \frac{n\sin(n)}{n^3+1} \cdot n^2 = \lim_{n\to\infty} \frac{n^3\sin(n)}{n^3+1}$$ does not exist.

Example.

We can use simple comparision test:

Notice $|\sin n| < 1$

so, $0 < \left|\frac{n \sin n}{n^3+1}\right| < \frac{n}{n^3+1}$

we know $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$ converges by integral test

we get that $\sum_{n=1}^{\infty} \left|\frac{n \sin n}{n^3+1}\right|$ converges by comparision test. So, $\sum_{n=1}^{\infty} \frac{n \sin n}{n^3+1}$ is absolutely convergent.