Application of integration.

Work. Energy expended acting againts a force.

eg: energy expended moving a weight against gravity.

Definition:
- Time t - measured in seconds.
- Mass m - measured in kg.
- position s - measured in meters

Newton's second law of motion.

Force = mass × acceleration i.e. $F = m\dfrac{d^2s}{dt^2}$

Newton = kg m/s^2.

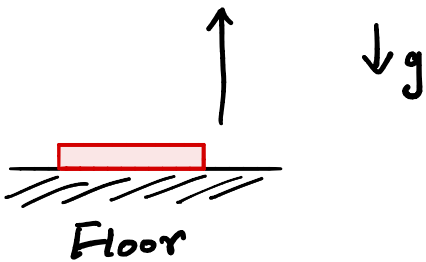
Work at constant force

Work = Force × displacement = $F \times d$

Joules = $N\cdot m$

Question (Constant Force)

How much work done moving a 1 kg book from floor to top of 2m high shelf?



Displacement = 2m

acceleration due to gravity = $9.8\ m/s^2$

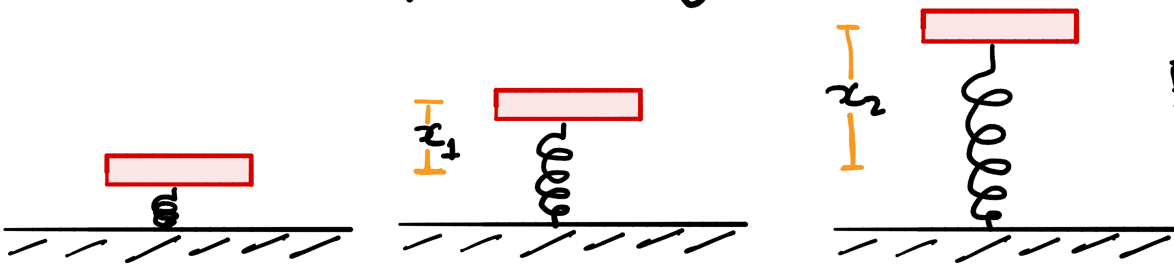
Force = $m \cdot a$ = 9.8 N

Work = $F \cdot d$ = 19.6 J.

Easy!

Variable force

$F(x)$ is the force acting at position x.

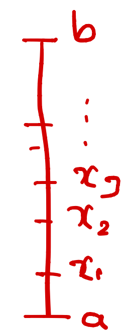


$F(x_1) > F(x_2)$

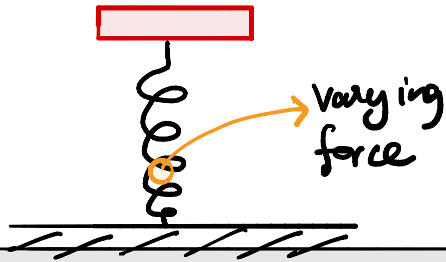
Use Riemann sum to approx. work done to move from $x=a$ to $x=b$.

- n subintervals $[x_{i-1}, x_i]$, $x_i = a + i\overbrace{\frac{(b-a)}{n}}^{\Delta x}$
- Constant force $[x_{i-1}, x_i]$, $F(x_i^*)$
- Approximate work, $R_n = \sum_{i=1}^{n} F(x_i^*)\Delta x$.
- $\text{Work} = \int_a^b F(x)\,dx$

$W = F(b-a)$



Question (variable force)



Hooke's law

Force $= k \cdot x$

k - spring constant

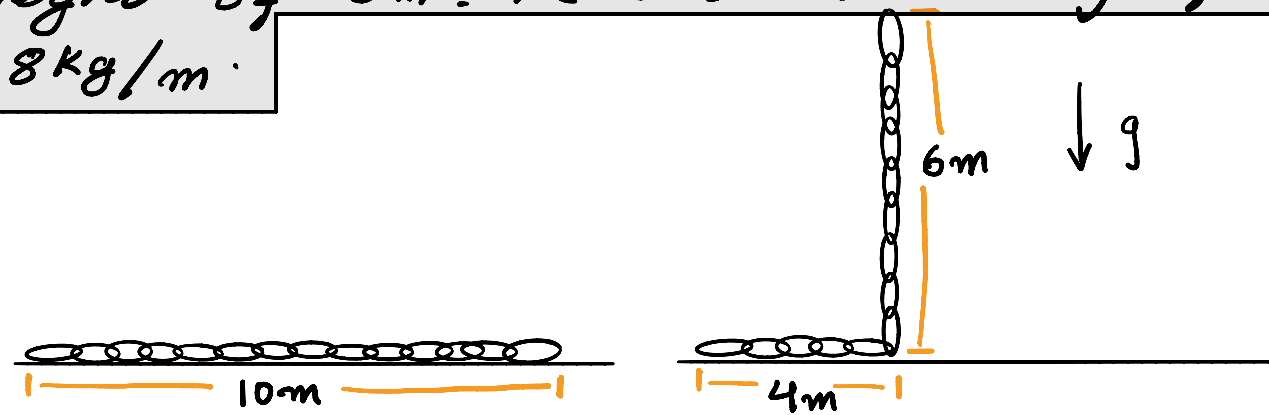
x - amount of stretching.

A spring has a natural length of 20cm. If a 25 N force is required to keep it stretched at a length of 30 cm, how much work is required to stretch it from 20 cm to 25cm?

- Spring constant : $F = k \cdot x \Rightarrow 25 = k(0.3 - 0.2)$
  $\Rightarrow k = 250\ N/m$
- Work $= \int_0^{0.05} F(x)\,dx = \int_0^{0.05} 250x\,dx = 125x^2 \Big|_0^{0.05} = 0.313J.$

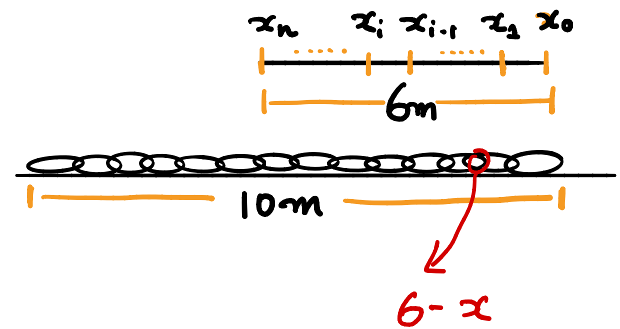
Example

A chain lying on the ground is 10 m long and weighs 80 kg. How much work is required to raise one end of the chain to height of 6 m? The constant density of the chain is 8 kg/m.



Riemann sum: split chain into segments $[x_{i-1}, x_i]$ and figure out how much work is done in lifting each segment.

Consider segment $[x_{i-1}, x_i]$:



- Piece of chain at x is lifted to $6-x$ m. Hence, segment $[x_{i-1}, x_i]$ is lifted to $6-x_i^*$ m.
- mass $=$ density $\cdot \Delta x = 8\Delta x$
- Segment experiences $(8\Delta x)\cdot 9.8 = 78.4\Delta x$ force.
- Riemann sum: $R_n = \sum_{i=1}^{n} (6-x_i^*)\, 78.4\Delta x$
- $n \to \infty$: $W = 78.4\int_0^6 (6-x)\,dx = 78.4\left[6x - \frac{x^2}{2}\right]_0^6$

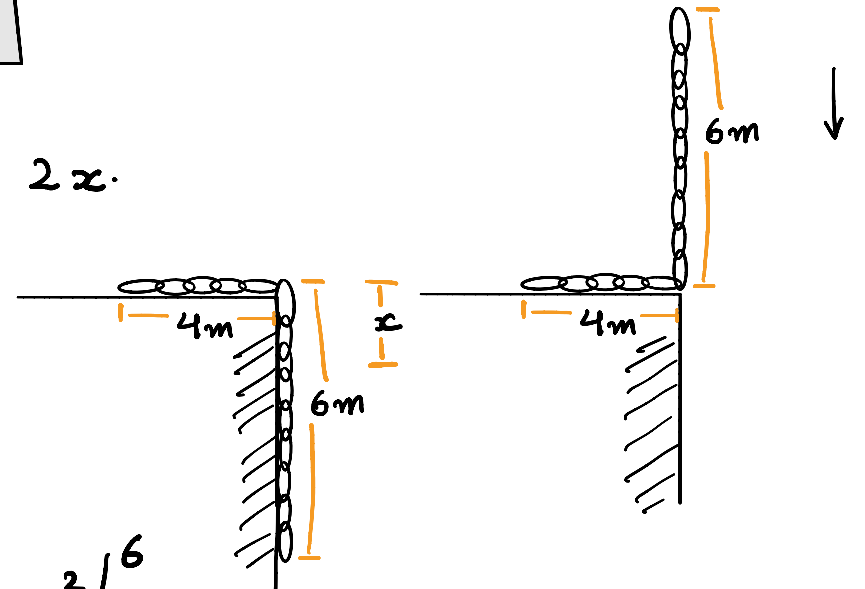
$= 1411.2$ J.

check.

Example.

What if the chain was dangling from the roof and we were to lift the far end?

- Let x be the distance
- A piece of chain is move $2x$.



So, Riemonn sum is

$$W \approx \sum_{i=1}^{n} 2x_i^* \, 78.4 \, \Delta x$$

So, work done is

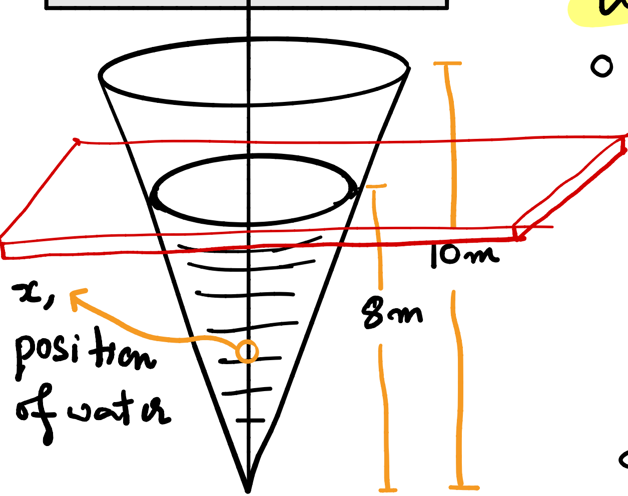
$$W = \int_0^6 156.8 \, x \, dx = 156.8 \left. \frac{x^2}{2} \right|_0^6$$

$$= 156.8 \times 18$$

$$= 2822.4 \text{ J}.$$

Example

A tank shaped like inverted cone of height 10m and radius 4m is filled to height of 8m. Assume density of water is 1000 kg/m³. Find work of pumping all water out of tank.



Work for each slice of water:

- mass = volume × density

Δm = mass in slice

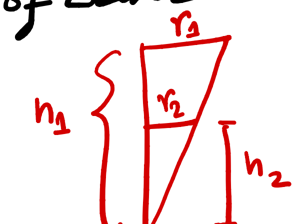
$= \pi(r(x))^2 \Delta x \times 1000$

$r(x)$ is radius of cross. section at x

$r(x) = \frac{2}{5}x.$

- Force acting on slice of water

$F = 9.8\,\Delta m$



$\frac{r_1}{r_2} = \frac{h_1}{h_2}$

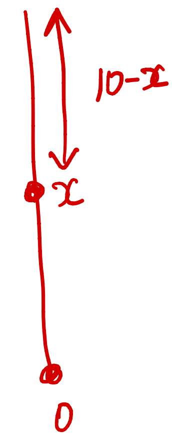
- Work for slice: $\Delta W = (10-x)\, 9.8\, \Delta m$

$$= (10-x)\, 9.8 \pi \left(\frac{4}{25} x^2\right) \Delta x\, 1000$$

So, $$W \approx \sum_{i=1}^{n} (10-x)\, 9.8 \pi \left(\frac{4}{25} x^2\right) 1000\, \Delta x$$

$$\Rightarrow W = \frac{9.8\pi \times 1000 \times 4}{25} \int_0^8 (10-x)\, x^2\, dx$$

$$= 3.36 \times 10^6 \text{ Joules.}$$



Work and Energy

Assume we are moving against a force, $F(x)$.
Assume position is given by $x(t)$, where t is time.

$$\text{Work done} = \int_a^b F(x)\,dx = \int_{t=\alpha}^{t=\beta} F(x(t))\,\frac{dx}{dt}\,dt.$$

By $F=ma$:

$$\text{Work done} = \int_\alpha^\beta m\,\frac{d^2x}{dt^2}\cdot\frac{dx}{dt}\cdot dt = m\int_\alpha^\beta v'(t)\,v(t)\,dt$$

$$= m\int_\alpha^\beta \frac{d}{dt}\left(\frac{1}{2}v(t)^2\right)dt$$

$$= m\left[\frac{1}{2}v(t)\right]_\alpha^\beta$$

$$= \underbrace{\frac{1}{2}m\,v(\beta)^2}_{\text{Kinetic energy.}} - \frac{1}{2}m\,v(\alpha)^2$$

Work done is difference in Kinetic energy

$x(\alpha)=a$
$x(\beta)=b$

$$\int_a^b f(x)\,dx.$$

$x(\alpha) = a$

$x(\beta) = b.$

$\underbrace{x = x(t)}$

Let