Quiz 2 solutions

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Solution 2:

We need to find the area between the curves $x = y^2 - 2$ and $x = e^y$ assuming a small segment dy along the $y - axis$.

$$\therefore \text{Required area} = \text{Red area - Blue area}$$

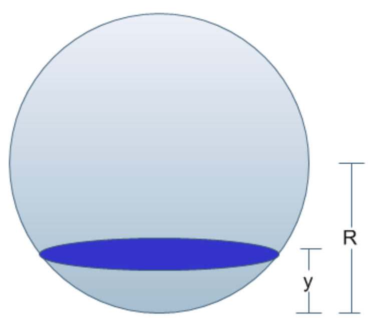
$$\implies \text{area} = \int_{-1}^{1} (e^y - y^2 + 2)\mathrm{d}y$$

$$\implies \text{area} = (e^y - \frac{y^3}{3} + 2y)\Big|_{-1}^{1}$$

$$\implies \text{area} = e - \frac{1}{e} + \frac{10}{3}$$

which is the final answer.

Solution 3:



The goal is to find the volume of liquid needed to fill a sphere of radius R to the height of $\frac{R}{3}$. Let us consider a thin horizontal disc of radius r at height y as shown in the figure.

$$\therefore \text{the volume will be } V = \int_{0}^{\frac{R}{3}} A(y)dy$$

where $A(y)$ is the area of the disc at height y.

Since area is just πr^2, therefore, we need to know r (which changes with y). Using Pythagorean Theorem,

$$R^2 = r^2 + (R-y)^2 \implies r^2 = R^2 - (R-y)^2 = 2Ry - y^2$$

and hence $A(y) = \pi(2Ry - y^2)$. Therefore, the volume needed to fill the sphere to height $\frac{R}{3}$ is

$$V = \int_0^{\frac{R}{3}} A(y)dy = \pi \int_0^{\frac{R}{3}} (2Ry - y^2)dy = \pi(Ry^2 - \frac{y^3}{3})\Big|_0^{\frac{R}{3}} = \pi(\frac{R^3}{9} - \frac{R^3}{81}) = 8\pi\frac{R^3}{81}$$

which will be the final answer.

Solution 4:

$$\int_2^6 t^3 \ \ln(5t)dt$$

We will use integration by parts to solve this. Let

$$u = \ \ln(5t) \text{ and } \ v'(t) = t^3$$

$$\int_2^6 t^3 \ \ln(5t)\mathrm{d}t = \frac{t^4}{4} \ \ln(5t)\Big|_2^6 - \int_2^6 \frac{t^4}{4} \cdot \frac{1}{t}\mathrm{d}t$$
$$\implies \frac{6^4}{4} \ \ln(5\cdot 6) - \frac{2^4}{4} \ \ln(5\cdot 2) - \frac{t^4}{16}\Big|_2^6$$
$$\implies \frac{6^4}{4} \ \ln(30) - \frac{2^4}{4} \ \ln(10) - \frac{6^4}{16} + \frac{2^4}{16}$$
$$\implies 324 \ \ln(30) - 80 - 4 \ \ln(10)$$

which will be the final answer.

Solution 5:

$$I = \int \frac{-8}{\sqrt{x^2+16}}\mathrm{d}x$$
$$I = -8\int \frac{1}{\sqrt{x^2+16}}\mathrm{d}x$$

We will use trigonometric substitution to solve this problem. Let $x = 4 \ \tan\theta$. $\therefore \mathrm{d}x = 4 \cdot \sec^2\theta \ \mathrm{d}\theta$.

Using this substitution , we get

$$I = -8\int \frac{1}{\sqrt{16\tan^2\theta + 16}} \cdot 4 \ \sec^2\theta \ \mathrm{d}\theta$$

$$I = \frac{-8\cdot 4}{4}\int \frac{1}{\sqrt{\tan^2\theta + 1}} \cdot \sec^2\theta \ \mathrm{d}\theta$$

$$I = -8\int \frac{1}{\sqrt{\sec^2\theta}} \cdot \sec^2\theta \ \mathrm{d}\theta \qquad \because \sec^2\theta = 1 + \tan^2\theta$$

$$I = -8 \int \sec\theta \, \mathrm{d}\theta$$

$$I = -8 \cdot \ln(|\tan\theta + \sec\theta|) + C$$

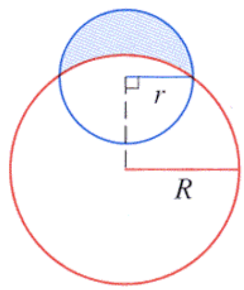
Using the Pythagoras theorem,

$$I = -8 \cdot \ln(|\frac{x}{4} + \frac{\sqrt{x^2+16}}{4}|) + C$$

$$I = -8 \cdot \ln(|x + \sqrt{x^2+16}|) + C$$

which is the final answer.

Solution 6:



The goal is to find the area in the shaded region. Assume, without loss of generality, that the bigger circle is centered at $(0,0)$. So, the smaller circle is centered at $(0,h)$, where $h = \sqrt{R^2 - r^2}$. The area in the shaded region is given by

$$\text{Shaded area} \ = \int_{-r}^{r} (\text{top} - \text{bottom}) dx, \tag{1}$$

where the top is given by the equation $y_{\text{top}} = \sqrt{r^2 - x^2} + h$ and the bottom is given by $y_{\text{bottom}} = \sqrt{R^2 - x^2}$. So, we have

$$\begin{aligned} \text{Shaded area} \ &= \int_{-r}^{r} (\sqrt{r^2 - x^2} + h - \sqrt{R^2 - x^2}) dx \\ &= \int_{-r}^{r} (\sqrt{r^2 - x^2} + \sqrt{R^2 - r^2} - \sqrt{R^2 - x^2}) dx \\ &= \underbrace{\int_{-r}^{r} \sqrt{r^2 - x^2} dx}_{I} + \underbrace{\int_{-r}^{r} \sqrt{R^2 - r^2} dx}_{II} - \underbrace{\int_{-r}^{r} \sqrt{R^2 - x^2} dx}_{III} \end{aligned}$$

Notice that integrals I and III are similar. We solve I using trigonometric substitution of $x = r\sin\theta$. So, $dx = r\cos\theta d\theta$ and

$$
\begin{aligned}
\int \sqrt{r^2 - r^2 \sin^2\theta}\; r\cos(\theta) d\theta = \int r^2 \cos^2(\theta) d\theta & \\
=& \frac{r^2}{2} \int \cos(2\theta) + 1 \; d\theta \qquad (2) \\
=& \frac{r^2}{2} \left(\frac{1}{2}\sin(2\theta) + \theta \right) + c \\
=& \frac{r^2}{2} \left(\sin(\theta)\cos(\theta) + \theta \right) + c, \qquad (3)
\end{aligned}
$$

where (2) holds because $\cos^2(\theta) = \frac{\cos(2\theta)+1}{2}$ and (3) holds because $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$. Now note that $\sin(\theta) = \frac{x}{r}$ and $\cos(\theta) = \frac{\sqrt{r^2-x^2}}{r}$. So, we have

$$
\begin{aligned}
I =& \left[\frac{r^2}{2} \left(\frac{x}{r} \frac{\sqrt{r^2 - x^2}}{r} + \arcsin(\frac{x}{r}) \right) \right]_{-r}^{r} \\
=& \left[\frac{r^2}{2} \left(\arcsin(1) - \arcsin(-1) \right) \right] \\
=& \frac{\pi r^2}{2}
\end{aligned}
$$

Similarly, we have

$$
\int \sqrt{R^2 - x^2}\; dx = \frac{R^2}{2} \left(\frac{x}{R} \frac{\sqrt{R^2 - x^2}}{R} + \arcsin\left(\frac{x}{R}\right) \right) + c
$$

and

$$
\begin{aligned}
III =& \left[\frac{R^2}{2} \left(\frac{x}{R} \frac{\sqrt{R^2 - x^2}}{R} + \arcsin(\frac{x}{R}) \right) \right]_{-r}^{r} \\
=& \left[\frac{R^2}{2} \left(\frac{r}{R} \frac{\sqrt{R^2 - r^2}}{R} + \arcsin(\frac{r}{R}) - \frac{-r}{R} \frac{\sqrt{R^2 - r^2}}{R} - \arcsin(\frac{-r}{R}) \right) \right] \\
=& \frac{R^2}{2} \left(\frac{2r}{R} \frac{\sqrt{R^2 - r^2}}{R} + 2\arcsin(\frac{r}{R}) \right) \\
=& r\sqrt{R^2 - r^2} + R^2 \arcsin(\frac{r}{R}).
\end{aligned}
$$

So, the shaded area is

$$
\begin{aligned}
\text{Shaded area } &= I + II - III \\
&= \frac{\pi r^2}{2} + 2r\sqrt{R^2 - r^2} - r\sqrt{R^2 - r^2} - R^2 \arcsin(\frac{r}{R}) \\
&= \frac{\pi r^2}{2} + r\sqrt{R^2 - r^2} - R^2 \arcsin(\frac{r}{R})
\end{aligned}
$$