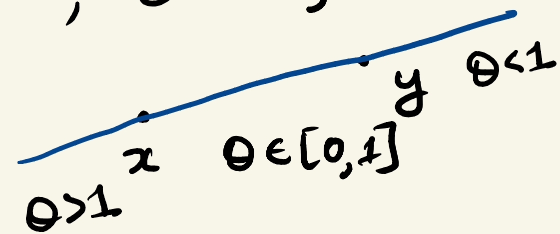
Convex sets

- Affine sets
- Convex sets
- Examples.

Affine sets.

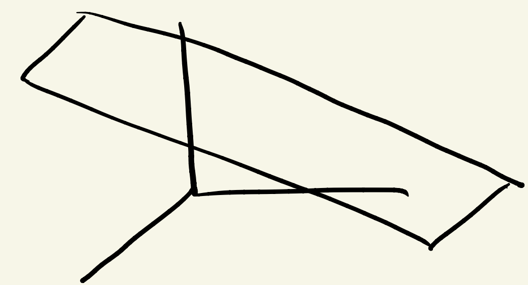
A line through the distinct points $x, y \in \mathbb{R}^n$

$$S = \{ z \mid z = \theta x + (1-\theta) y \ , \ \theta \in \mathbb{R} \}$$



Affine set contains all line through any distinct point in the set.

$$x, y \in S \iff \theta x + (1-\theta) y = z \in S, \quad \forall \theta \in \mathbb{R}$$



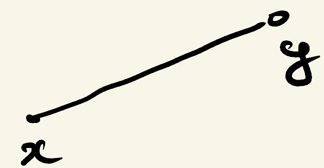
Example: The solution set of linear equations:
$$S = \{ x \mid Ax = b \}.$$

$x, y \in S$, $\theta x + (1-\theta) y = z$ satisfies $Az = b$.

Convex combination.

line segment between any two points $x, y \in \mathbb{R}^n$ is:

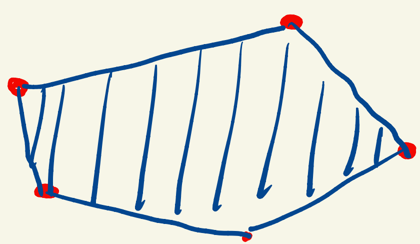
$$z = \theta x + (1-\theta) y \ , \ 0 \le \theta \le 1$$



x is a convex combination of $x_1, \dots, x_n$ if

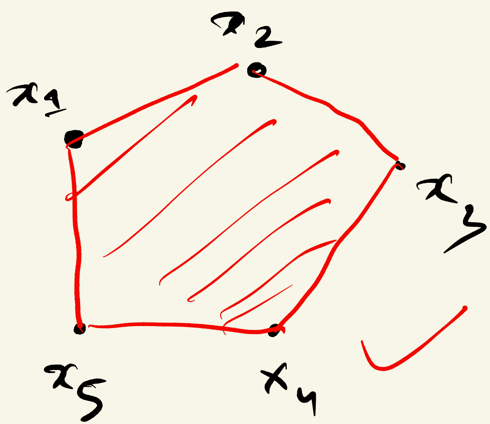
$$x = \theta_1 x_1 + \theta_2 x_2 + \dots + \theta_n x_n = \sum_{i=1}^{n} \theta_i x_i \quad \text{and}$$

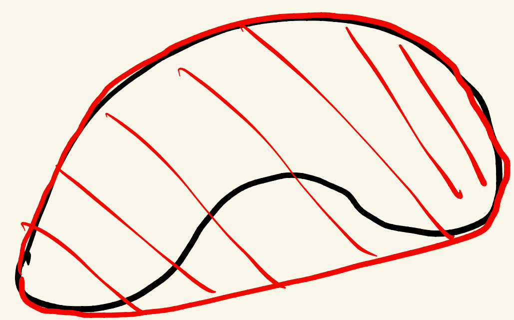
$$\sum_{i=1}^{n} \theta_i = 1 \quad \text{with} \quad \theta_i \ge 0.$$



The convex hull of S contains all convex combinations of points S:

$$\operatorname{co} S = \left\{ z = \sum_{i=1}^{n} \theta_i x_i \;\middle|\; x_i \in S,\ \sum_{i=1}^{n} \theta_i = 1,\ \theta_i \geq 0 \right\}$$



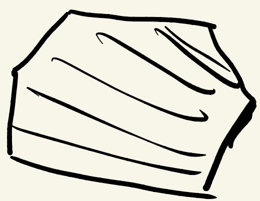


Convex set:

$S \in \mathbb{R}^n$ is a convex if it contains all convex combinations of points in S.

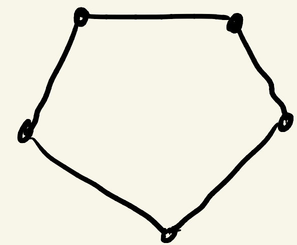
$$x, y \in S \Leftrightarrow 0 \leq \theta \leq 1, \quad \theta x + (1-\theta) y \in S.$$

<u>Example:</u>



convex

not-convex



not-convex.

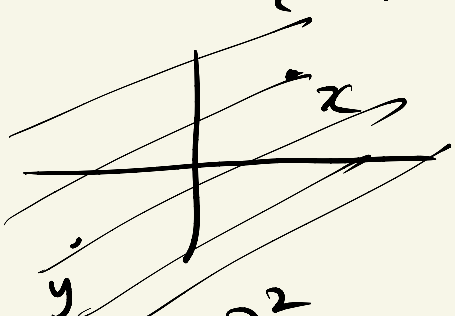
Linear set:

S is a subspace if

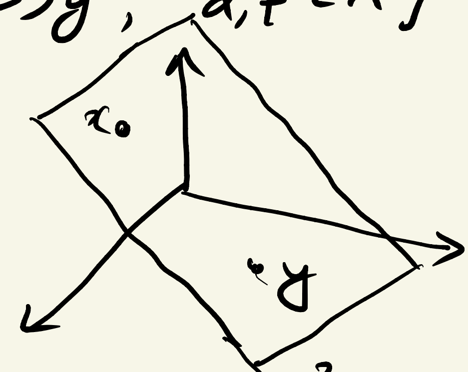
- $0 \in S$
- closed under addition
- closed under scalar multiplication.

S is the subspace containing two distinct points $x, y \in \mathbb{R}^n$

$$S = \{ z \mid z = \alpha x + (1-\beta) y, \ \alpha, \beta \in \mathbb{R} \}.$$



$\mathbb{R}^2$

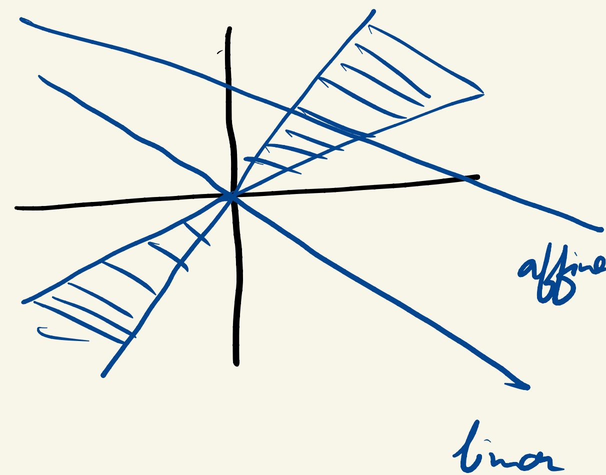


$\mathbb{R}^3$

A linear set contains all lines through distinct point in the set:

$$x \in S,\ y \in S \iff \alpha x + \beta y = z \in S, \quad \forall \alpha, \beta \in \mathbb{R}.$$

Linear sets are affine but affine sets are not always linear.



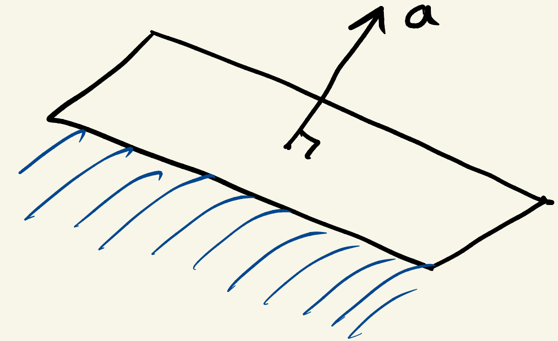
Example: The range and null space of a matrix is a linear set.

Half space and hyperplanes

hyperplane: set of the form $\{x \mid a^T x = b\}$.

half space: set of the form

$$\{x \mid a^T x \le b\}.$$



- $a \neq 0$ is a normal vector.
- hyperplane are affine and convex.

pick $x, y \in S$, $a^T[\theta x + (1-\theta) y]$

$$= \theta b + (1-\theta) b$$

$$= b \quad \text{if } \theta \in [0,1].$$

$$\Rightarrow \theta x + (1-\theta) y \in S.$$

- half space are convex but are not affine.
- Example: The non-negative orthant

$$R_+^n = \{ x : x_i \geq 0 \text{ for } i = 1, \dots, n \}.$$

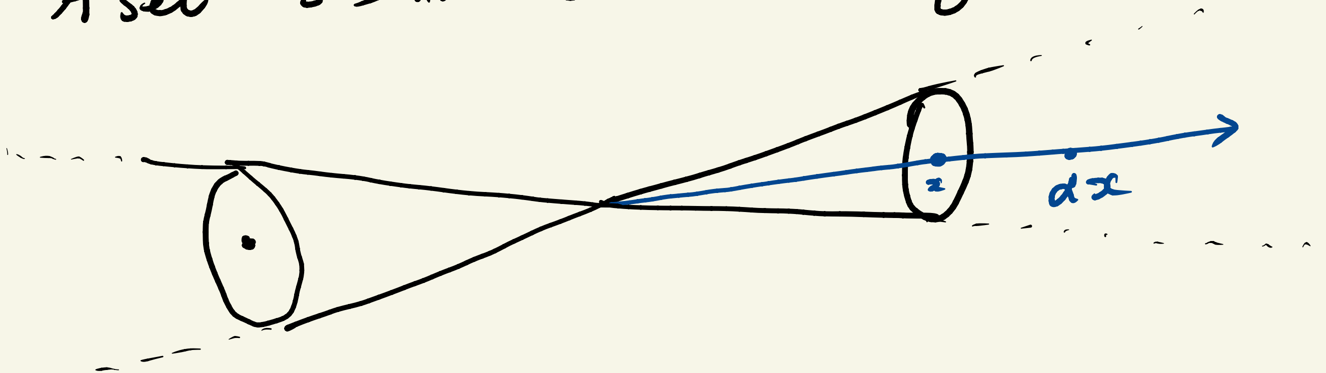
is intersection of halfspace

$$H_i = \{ x \in \mathbb{R}^n / \ e_i^T x \geq 0 \}.$$

- The non-negative orthant is a cone

Cone

A set $S \subset \mathbb{R}^n$ is a cone if $x \in S \iff \alpha x \in S \quad \forall \alpha \geq 0$

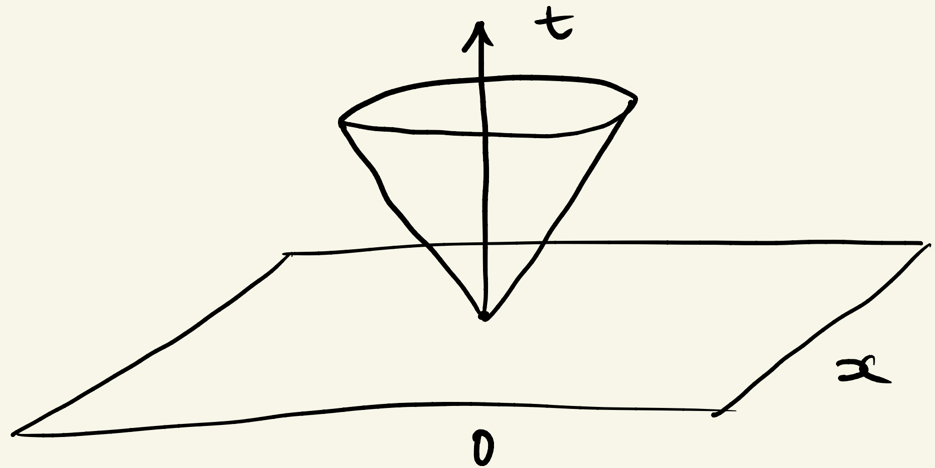


A convex cone is a cone that is convex:

$$x, y \in S \iff \underbrace{\theta_1 x + \theta_2 y}_{\text{conic combination.}} \in S, \quad \forall \theta_1, \theta_2 \geq 0.$$

Example: $\mathbb{R}^n_+$ - non-negative orthant

$L^n_+ - \left\{ \begin{bmatrix} x \\ t \end{bmatrix} \middle| \ \|x\| \leq t, \ x \in \mathbb{R}^n, t \in \mathbb{R}_+ \right\}$



t
x
0