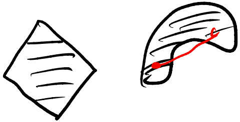
14. Convex functions and problems

- affine sets
- convex sets
- examples

Previous lecture

Convex set: A set S is convex if for any points $x, y \in S$ and $\lambda \in [0,1]$ the point $\underbrace{\lambda x + (1-\lambda) y}_{\text{line segment}} \in S$.



Convex Hull: The convex Hull of S contains all convex combinations of points in S.

$$\operatorname{conv}(S) = \left\{ x \,\middle|\, x = \sum_{i=1}^{k} \lambda_i x_i,\ x_i \in S,\ \lambda_i \geq 0,\ \sum_{i=1}^{k} \lambda_i = 1,\ k \in \mathbb{N} \right\}$$

$\operatorname{co}(S)$

Euclidean ball and ellipsoid

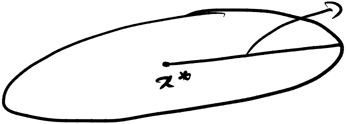
Euclidean ball centred at x^* with radius r:

$$\mathcal{B}(x^*, r) = \{x \mid \|x - x^*\|_2 \leq r\}$$

$$= \{x^* + r u \mid \|u\|_2 \leq 1\}$$

Ellipsoid: Let $P \succ 0$. The set:

$$S = \{x \mid (x - x^*)^T P (x - x^*) \leq 1\}$$



along eigen direction with smallest eigenvalue

Alternatively:

$$S = \{x^* + P u \mid \|u\|_2 \leq 1,\ P \succ 0\}$$

Norm balls

Norm: A function $\|\cdot\|$ is a norm if it satisfies:

- $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$
- $\|\alpha x\| = |\alpha|\|x\|$ for any $\alpha \in \mathbb{R}$
- $\|x + y\| \leq \|x\| + \|y\|$

Any norm ball with centre x^* and radius r:

$$B(x^*, r) = \{x \,|\|x - x^*\| \leq r\}$$

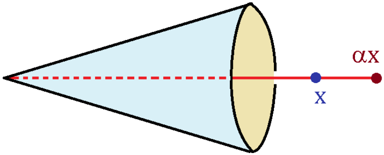
Proof: Let $x, y \in B(x^*, r)$, $\lambda \in [0,1]$

Let $z = \lambda x + (1-\lambda) y$

$$\begin{aligned}\|z - x^*\| &= \|\lambda x + (1-\lambda) y - x^*\| \\ &= \|\lambda(x - x^*) + (1-\lambda)(y - x^*)\| \\ &\leq \lambda \|x - x^*\| + (1-\lambda)\|y - x^*\| \\ &\leq \lambda r + (1-\lambda) r = r\end{aligned}$$

Convex cones

A set $\mathcal{S} \subseteq \mathbf{R}^n$ is a **cone** if $x \in \mathcal{S} \iff \alpha x \in \mathcal{S}\ \forall \alpha \geq 0$



A set $\mathcal{S}$ is a **convex cone** if

1. $x, y \in S \Rightarrow x + y \in S.$
2. $x \in S,\ \lambda \geq 0 \Rightarrow \lambda x \in S.$

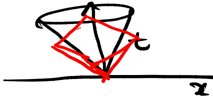
conic combination.

A convex cone contains **conic combinations** of its elements

$$x, y \in \mathcal{S} \iff \theta_1 x + \theta_2 y \in \mathcal{S},\ \forall \theta_1, \theta_2 \geq 0$$

Given k points $\{x_i\}_{i=1}^k$, a conic combination of $\{x_i\}_{i=1}^k$ is $x = \sum_{i=1}^{k} \lambda_i x_i$, $\lambda_i \geq 0$.

Examples of convex cone



- Non-negative orthant:

$$\mathbb{R}_+^n = \{ x \mid x_i \geq 0, \quad i = 1, \dots, n \}$$

- Second order cone

$$\mathcal{L}_+^n = \left\{ \begin{bmatrix} x \\ t \end{bmatrix} \in \mathbb{R}^{n+1} \;\middle|\; \|x\|_2 \leq t, \; \begin{matrix} x \in \mathbb{R}^n \\ t \in \mathbb{R}_+ \end{matrix} \right\}$$

$t \geq 0$

- Any norm-cone:

$$S = \left\{ \begin{bmatrix} x \\ t \end{bmatrix} \in \mathbb{R}^{n+1} \;\middle|\; \|x\| \leq t, \; \begin{matrix} x \in \mathbb{R}^n \\ t \in \mathbb{R}_+ \end{matrix} \right\}$$

- Positive semi-definite cone:

$$S_+^n = \{ X \mid u^T X u \geq 0, \; u \in \mathbb{R}^n \}$$

S_{++}^n – set of positive definite matrices

Operations that preserve convexity

Linear combinations Let S_1, S_2 be convex and $\alpha_1, \alpha_2 \in \mathbb{R}$ Then

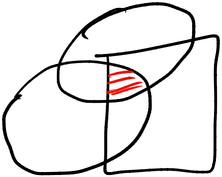
$$\alpha_1 S_1 + \alpha_2 S_2 = \{\alpha_1 s_1 + \alpha_2 s_2 \mid s_1 \in S_1, s_2 \in S_2\}$$

is convex.

Intersections

$S_1, \dots, S_k$ are convex

$\Rightarrow \quad S_1 \cap \dots \cap S_k$ is convex.



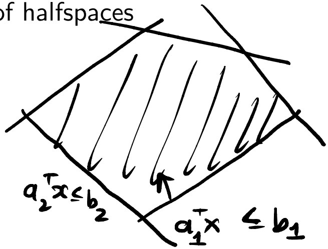
Linear Maps:

$A \in \mathbb{R}^{m \times n}, \quad S \subseteq \mathbb{R}^n$, S- convex.

Then $\quad A(S) = \{As \mid s \in S\}$ is convex.

Convex Polytopes

$\mathcal{S}$ is a **convex polytope** if it is the intersection of halfspaces



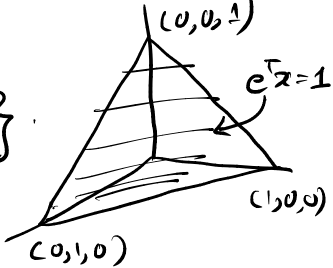
$$S = \bigcap_{i=1}^{n} \{ x \mid a_i^T x \le b_i \}$$

$$= \{ x \mid Ax \le b \}$$

$$A = \begin{bmatrix} - a_1^T - \\ \vdots \\ - a_n^T - \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

Example:

$$S = \left\{ x \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^{n} x_i = 1, \; x_i \ge 0 \right\}$$



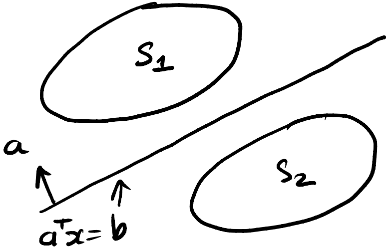
Separating hyperplane theorem

If S_1 and S_2 are non-empty and disjoint convex sets.

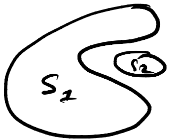
There exists $a \neq 0$ and $b \in \mathbb{R}$ s.t.

① $a^T x \leq b$ for all $x \in S_1$

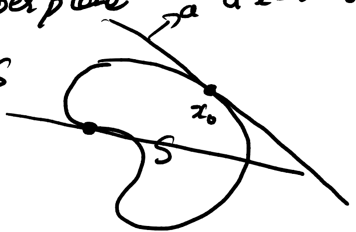
② $a^T x \geq b$ for all $x \in S_2$.



The hyperplane $\{x \mid a^T x = b\}$ is the separating hyperplane.



Supporting hyperplane



The supporting hyperplane for S at x_0 at boundary of S is

$$\{x \mid a^Tx = a^Tx_0\}$$

with $a \neq 0$, $a^Tx \leq a^Tx_0$ for all $x \in S$.

Supporting hyperplane theorem:

Every boundary point of a convex set has a supporting hyperplane.