17. Reduced Gradient

- Newton's method for linear constraint.

Previous lecture

Sufficient condition for optimality of x^* for

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t. } Ax = b.$$

f - smooth, $A \in \mathbb{R}^{m \times n}$, $m \leq n$

x^* is a strict local minimizer if

① Feasibility : $Ax^* = b$

② Optimality : $\nabla f(x^*) = A^T y$ for some $y \in \mathbb{R}^m$.

③ Positivity : $p^T \nabla^2 f(x^*) p > 0$ for all $p \in N(A) \setminus \{0\}$.

Equality constrained with quadratic min.

$$\min_{x \in \mathbb{R}^n} \ \frac{1}{2} x^T P x + q^T x + r \quad \text{s.t.} \quad Ax = b,$$

where $P \succeq 0$ and $A \in \mathbb{R}^{m \times n}$.

Optimality Conditions:

① Feasibility : $Ax^* = b$

② Optimality : $Px^* + q = A^T y^*$

$\Big\}$ KKT system.

System of Equation in $n+m$ variables:

$$\begin{bmatrix} -P & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} q \\ b \end{bmatrix}$$

KKT matrix

Equality constrained with quadratic min. (contd.)

- If the KKT system is not solvable, the constrained quadratic opt. problem is unbounded below or infeasible.
- The KKT matrix is non-singular if any of the following holds:

① $\mathcal{N}(A) \cap \mathcal{N}(P) = \{0\}$.

② For any $x \neq 0$, $Ax = 0 \Rightarrow x^T P x > 0$. ($P$ is PD on $\mathcal{N}(A)$)

③ $Z^T P Z > 0$, Z is a basis for $\mathcal{N}(A)$.

Newton's method for equality constraint

Consider

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad Ax = b.$$

basis for $N(A)$.

Let $\bar{x}$ be a particular solution and $R(Z) = N(A)$.

Quadratic approximation at $\bar{x}$:

$$f(\bar{x} + Zp) \approx f(\bar{x}) + \nabla f(\bar{x})^T (Zp) + \frac{1}{2} p^T Z^T \nabla^2 f(\bar{x}) Z p$$

We can solve:

$$\min_{p \in \mathbb{R}^{n-m}} \frac{1}{2} p^T \bar{H} p + \bar{g}^T p + \bar{f},$$

$$\bar{H} = Z^T \nabla^2 f(\bar{x}) Z, \quad \bar{g} = Z^T \nabla f(\bar{x}), \quad \bar{f} = f(\bar{x}).$$

Newton direction: $\bar{H} d = -\bar{g}$

Algorithm

Input: $\bar{x}$ feasible point and Z basis for $N(A)$

For $k = 1, 2, 3,$

1. Compute $g := \nabla f(x_k)$
2. Compute $H := \nabla^2 f(x_k)$
3. Solve $Z^T H Z d = -Z^T g$ to get d_k
4. Line search $f(x_k + \alpha Z d_k)$
5. $x_{k+1} = x_k + \alpha_k d_k$
6. Stop if converged.

Example

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{subject to} \quad Ax = b \quad \text{— } ①$$

- An approach to solving ① is to solve the system of equations corresponding to optimality condition (KKT system)

- Algorithm approach:
  1. Find particular solution $\bar{x}$ and a basis Z for null space of A (assume A full row rank)
  2. Solve the unconstrained problem

$$\min_{p \in \mathbb{R}^{n-m}} f(\bar{x} + p) \quad \underbrace{\text{s.t. } A(\bar{x} + Zp) = b}_{\text{Always satisfied.}}$$

using a descent method.

Basis of Null (A)

① Using QR- decomposition :

$$A^T = \begin{bmatrix} \hat{Q} & \bar{Q} \end{bmatrix} \begin{bmatrix} R \\ 0 \end{bmatrix}$$

$$R(\hat{Q}) \oplus R(\bar{Q}) = \mathbb{R}^n$$

- $R(A^T) = R(\hat{Q})$
- $N(A) = R(\bar{Q})$

Basis of Null (A)

Permute columns of A so, that

$$A = [B \quad N],$$

B - basic matix, nonsingular (Note that A is full rank)

N - non-basic matrix.

If x is feasible:

$$Ax = b \iff [B \quad N] \begin{bmatrix} x_B \\ x_N \end{bmatrix} = b \iff Bx_B + Nx_N = b$$

Consider $\quad Z = \begin{bmatrix} -B^{-1}N \\ I \end{bmatrix}, \quad AZ = 0.$