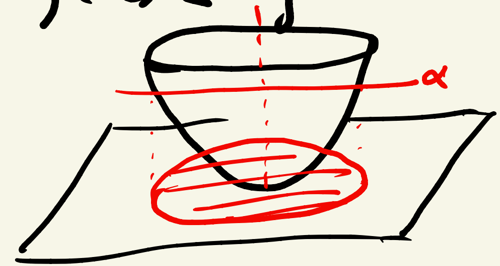
22. convex functions

- level set
- Epigraph
- Optimality for convex opt.

Level sets

- The level set of a function $f : S \to \mathbb{R}$ is a set

$$L_\alpha(f) = [f \leq \alpha] := \{x \in S \mid f(x) \leq \alpha\}.$$



- f is convex $\Rightarrow$ all level sets are convex.

proof:
- Take $x, y \in L_\alpha(f)$. Then $f(x) \leq \alpha$, $f(y) \leq \alpha$
- Because f is convex, for any $\theta \in [0,1]$.

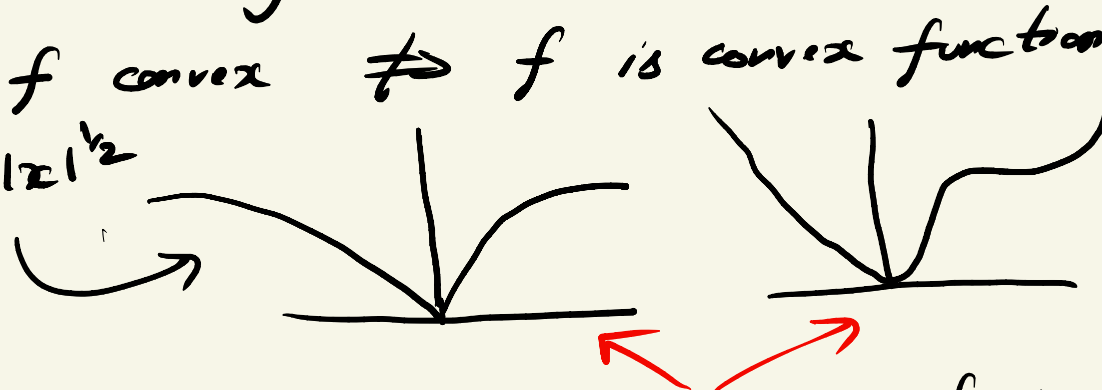
$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$$
$$\leq \theta \alpha + (1-\theta)\alpha = \alpha$$

- so, $f(\theta x + (1-\theta)y) \in L_\alpha(f) \Rightarrow L_\alpha(f)$ is convex.

Quasi-convex Functions

- Converse is not necessarily true:
  all levels of f convex $\not\Rightarrow$ f is convex function.

- eg: $f(x) = |x|^{1/2}$



- all level sets of f convex $\Rightarrow$ quasi-convex functions.

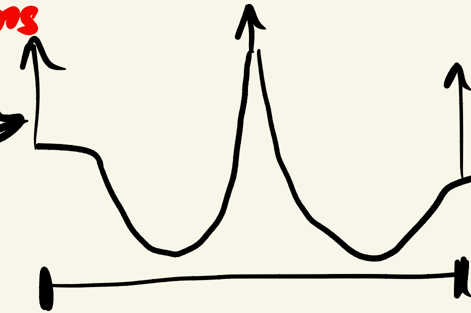
- f is quasi-concave if $-f$ is quasi-convex.

- f is quasi-linear if it is both quasi-convex and quasi concave.

Extended Real-Valued Functions

- Extend $f: S \to \mathbb{R}$ to $\tilde{f}: \mathbb{R}^n \to \mathbb{R}$ by

$$\tilde{f}(x) = \begin{cases} f(x) & x \in S \\ \infty & x \notin S. \end{cases}$$



- effective domain : $\operatorname{dom}(\tilde{f}) = \{x \in \mathbb{R}^n \mid \tilde{f}(x) < \infty\}$.
- Convexity: $\tilde{f}$ is convex if for any $x, y \in \mathbb{R}^n$, $\lambda \in [0,1]$.

$$\tilde{f}(\lambda x + (1-\lambda) y) \le \lambda \tilde{f}(x) + (1-\lambda) \tilde{f}(y).$$

$\iff$
- $\operatorname{dom}(\tilde{f})$ is convex
- for any $x, y \in \operatorname{dom}(\tilde{f})$ and $\lambda \in [0,1]$.

$$f(\lambda x + (1-\lambda) y) \le \lambda f(x) + (1-\lambda) f(y).$$

Epigraph

- $f: S \to \mathbb{R}$ defined over a set $S \subset \mathbb{R}^n$. extend to extended real function.

The epigraph of f is

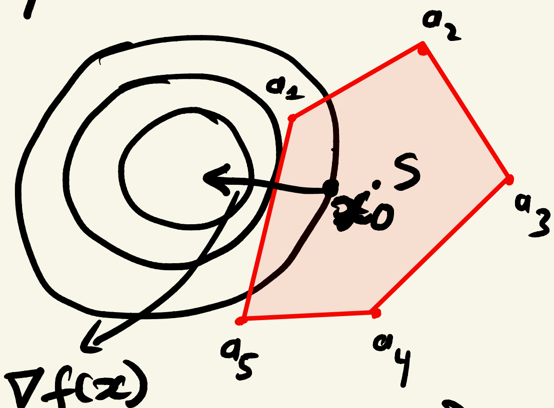
$$\mathrm{epi}(f) = \left\{ \begin{pmatrix} x \\ t \end{pmatrix} \in R^{n+1} \,\middle|\, x \in S,\ f(x) \le t \right\}.$$

- $\mathrm{epi}(f)$ is convex iff f is convex.

Support Function.

- Let $S \subseteq \mathbb{R}^n$. The support function of S at x is

$$\sigma_S(x) = \max_{y \in S} x^T y.$$



$S = \mathrm{conv}(\{a_i\})$

$\sigma_S(-\nabla f(x)) = a_1$

convex

- Let $f_i : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$, $i \in I$

where I is an arbitrary index set.

Then, the function

$$\max_{i \in I} f_i(x) \text{ is convex}$$

- Support function is convex (even if S is not convex).

Example

- $S = B_1(0) = \{ y \in \mathbb{R}^n \mid \|y\|_2 \leq 1 \}$.

$$\sigma_S(x) = \max_{y \in S} x^T y = \|x\|$$

If $x \neq 0$, then

$$x^T y \leq \|x\| \, \|y\| \leq \|x\|$$

and equality is achieved if $y = \dfrac{x}{\|x\|_2}$

- $S = \{ y \in \mathbb{R}^n \ \ \|y\|_1 \leq 1 \}$

$$\sigma_S(x) = \max_{y \in S} x^T y = \|x\|_\infty$$

Convex optimization problem

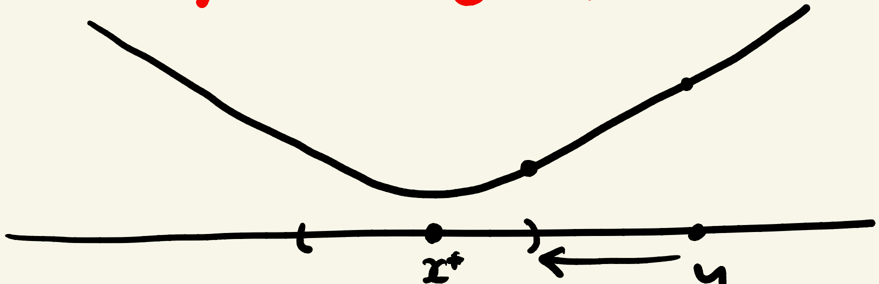
- Optimization problem with convex objective $f: \mathbb{R}^n \to \mathbb{R}$ and convex constraint set S.

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{subject to } x \in S.$$

- (Local min = global min). Let $x^* \in S$ be a (strict) local minimizer of f over S. Then x^* is a (strict) global minimizer.

- global minimizer $\Rightarrow f(x^*) \leq f(y) \quad \forall y \in S.$

Global optimality of local min



$\tilde{y} = \lambda y + (1-\lambda) x^*, \quad \lambda \in [0,1]$

proof:

- Local minimum $\Rightarrow \exists\, r > 0$ s.t. $f(x^*) \le f(y)$ for all $y \in B(x^*, r)$
- Let $y \in S$ and pick λ s.t. $\tilde{y} = \lambda y + (1-\lambda) x^* \in B(x^*, r)$
- $f(x^*) \le f(\lambda y + (1-\lambda) x^*) \le \lambda f(y) + (1-\lambda) f(x^*)$.
- Rearrange $\Rightarrow \quad \lambda f(x^*) \le \lambda f(y)$.

$\Rightarrow \quad f(x^*) \le f(y)$.

Convexity of optimality set

- The set of global minimizers of a convex opt problem is a convex set.

$$X^* = \{ x^* \in S \mid f(x^*) \le f(y),\ \forall y \in S \}.$$

- In addition, if f is strictly convex, X^* has at most one element.

Sufficient 1st order condition

- (Recall) $f: C \to \mathbb{R}$ continuously differentiable over convex set C.

  x^* local minimum $\Longrightarrow$ x^* is stationary point

  $\overset{\text{(def)}}{\Longrightarrow}$ $\nabla f(x^*)^T(y-x^*) \geq 0$ for all $y \in C$

  $\overset{\text{(def)}}{\Longrightarrow}$ $-\nabla f(x^*) \in N_S(x^*)$

- $f: C \to \mathbb{R}$ convex continuously diff. over convex set C.

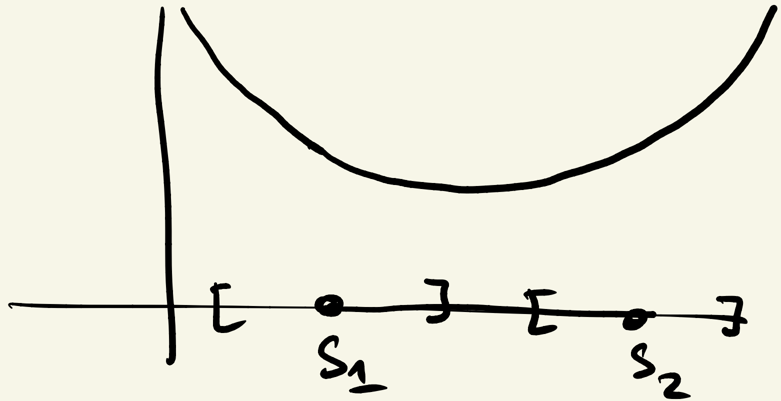
  x^* local minimum $\Longleftrightarrow$ x^* is stationary point

$$f(y) \geq f(x^*) + \nabla f(x^*)^T(y-x^*), \quad \forall y \in S.$$

$$\geq f(x^*).$$

Linear programming.

$$\min \quad c^T x \qquad \text{s.t.} \quad Ax = b$$
$$x \geq 0$$



$$S = S_1 \cup S_2$$

$$f(\underbrace{\qquad\qquad})$$