Linear programming

- problem setup
- extreme points

General linear program

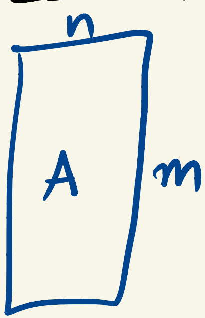
linear programming problem:

$$\min_{x \in \mathbb{R}^n} \; c^T x \qquad \text{subject to} \qquad Ax \le b.$$

$$\boxed{\begin{aligned} &\min \; c^T x \\ &\text{s.t. } Ax = b \\ &\qquad x \ge 0 \end{aligned}}$$

standard form.

- $c \in \mathbb{R}^n$ is a given cost vector.
- $x \in \mathbb{R}^n$ is unknown decision vector
- $A \in \mathbb{R}^{m \times n}$ is a given resource (measurement) matrix.
- $c^T x = \sum c_i x_i$ is a linear cost (objective) function.



LP formulation

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax \le b \end{aligned}$$

- Problem is infeasible:

  There is no $x \in \mathbb{R}^n$ such that $Ax \le b$.

- Problem is unbounded below:

  There exist some $x = x_0 + \alpha d$ where $Ax \le b$ for all $\alpha \ge 0$ and $c^T d < 0$

  x_0 feasible

  $$Ax = A(x_0 + \alpha d) = Ax_0 + \alpha Ad \le b + \alpha \underbrace{Ad}_{\le 0} \le b \quad \text{if } Ad \le 0$$

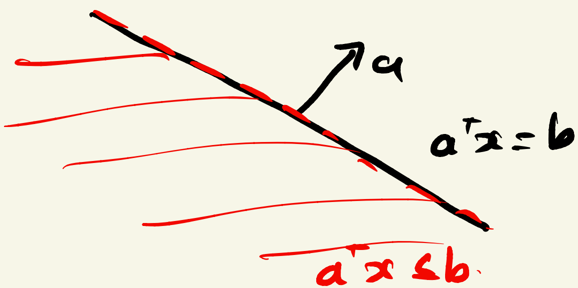
  want

  $$c^T(x_0 + \alpha d) = c^T x_0 + \alpha c^T d \to -\infty \quad \text{as} \quad \alpha \to +\infty$$

- Problem is feasible and has finite value

Polyhedra constraint

- A polyhedra is a set of the form $\{x \in \mathbb{R}^n / Ax \leq b\}$. $= \{x \in \mathbb{R}^n / a_i^T x \leq b_i, \ i=1,\dots,m\}$
  $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$.
- A polyhedra can be bounded or can extend to infinity:
  A set S is bounded if $S \subseteq B(0,R)$
  ball centered at 0 with radius R.
- Polyhedra are an intersection of half spaces and hyperplanes.



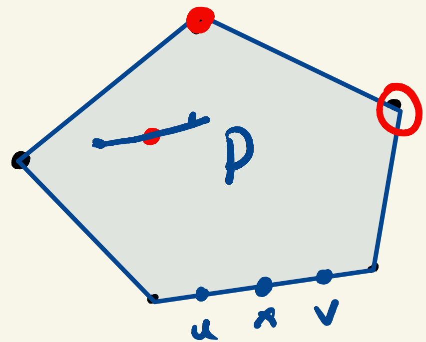
- Polyhedra are convex.

Extreme point

- Solution to LP tend to occur at a "corner" of a polyhedron.

extreme

Extreme points are geometric and does not use any representation of P directly.



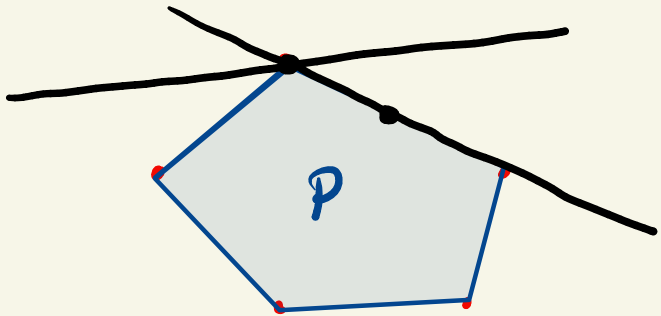
$x \in P$ is an extreme point if it is not a convex combination of points in P.

- For a polyhedron P, a vector $x \in P$ is an extreme point of P if we cannot find two vector $y, z \in P$, with $y \neq x$, $z \neq x$, and $\lambda \in [0, 1]$ such that

$$x = \lambda y + (1-\lambda) z$$

Vertex

- Alternate geometric definition of a corner of a polyhedron.



x is a vertex iff P is strictly on one side of a hyperplane through x.

- For a polyhedron P, $x \in P$ is a vertex if there exists a vector $c \in \mathbb{R}^n$ such that

$$c^T x < c^T y \quad \text{for all } y \in P,\ y \neq x.$$

Basic Feasible solution

- For a set of row indices $B \subseteq \{1, \dots, m\}$, A_B is the submatrix of A containing rows indexed in B.
- <u>Active constraint</u>: For any vector x^*, if $a_i^T x^* = b_i$ for some $i \in \{1, \dots, m\}$, we say the corresponding constraint is active.
- For $x^* \in \mathbb{R}^n$, let $I \subseteq \{1, \dots, m\}$ be index set containing active constraint (i.e. $I = \{i \mid a_i^T x^* = b_i\}$). TFAE:
  - $A_I x = b_I$ has a unique solution.
  - There exists n vectors in $\{a_i \mid i \in I\}$ that are linearly independent.
  - $\text{span}\{a_i \mid i \in I\} = \mathbb{R}^n$.

Basic Feasible solution

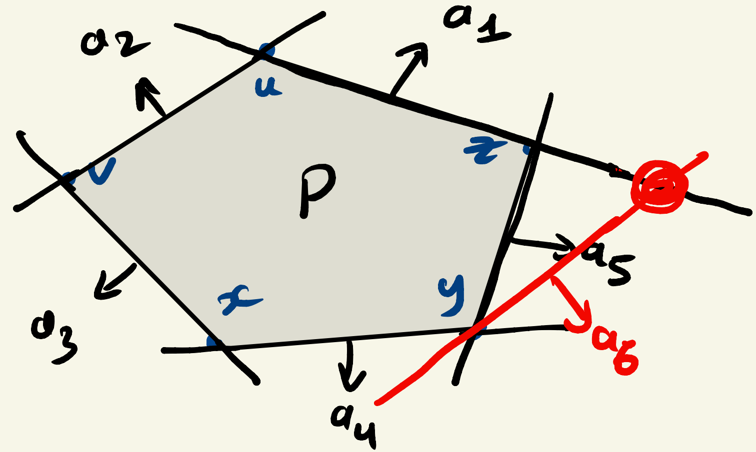
- For $x^* \in \mathbb{R}^n$, let $B \subseteq \{1, .., m\}$ be index set of active constraints
- x^* is a basic feasible solution of P if
  - $x^* \in P$ $\quad (Ax^* \leq b)$
  - A_B contains atleast n linearly independent rows.
- B is called basic set
- $N = \{1, .., m\} \setminus B$ is called non-basic set.

Basic feasible solution

$Ax \le b$

$a_i^T x \le b_i$

for $i = 1, \dots, 5$



$$\begin{aligned} a_1^T u &= b_1 \\ a_2^T u &= b_2 \end{aligned}$$

$a_3^T u < b_3$

$\vdots$

$$\begin{aligned} a_2^T v &= b_2 \\ a_3^T v &= b_3 \end{aligned}$$

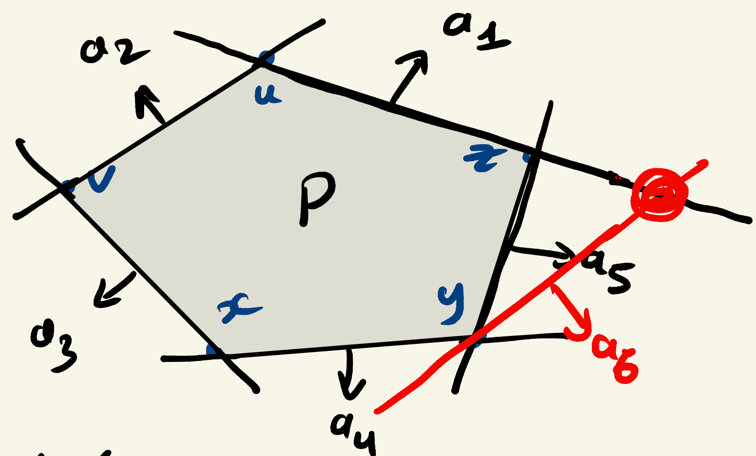
$a_4^T v < b$

$$\begin{aligned} a_3^T x &= b_3 \\ a_4^T x &= b_4 \end{aligned}$$

$\dots$

Degenerate solution.

- Pick a basic feasible solution x^*, with basic set B
- Then $\text{ran}(A_B) = n$ $\quad |B| \geq n$
  $|B| > n$
- If $\boxed{A_B \text{ is not invertible}}$, then x^* is a degenerate basic feasible solution



- y is a degenerate basic feasible solution

- Preprocess: Remove all redundant constraints.

Extreme point, vertex, BFS.

- For a polyhedron P, $x^* \in P$. TFAE:

  a) x^* is a vertex

  b) x^* is a extreme point

  c) x^* is a BFS.

$$B = \begin{bmatrix} -P & A^T \\ A & 0 \end{bmatrix} \rightarrow \text{KKT matrix}$$

B is invertible if

① $N(A) \cap N(P) = \{0\}$.

$$x \in N(B) \Rightarrow Bx = 0 = B \begin{bmatrix} x_p \\ x_A \end{bmatrix} = 0$$

$$\Rightarrow -Px_p + A^T x_A = 0$$

and $Ax_p = 0 \Rightarrow x_p \in N(A)$.

$S = \{x \mid Ax = b\}$

x-fixed

$S' = \{z - x \mid z \in S\} \subseteq N(A)$

Now, if $y \in N(A)$. want to show $y = z - x$ for

x-fixed and $z \in S$.

$$z = y + x$$

$$Az = Ay + Ax = Ax = b.$$

$$\Rightarrow z \in S.$$