__Recall:__

Vertex: $x^* \in P$ is vertex if $\exists\ c \in \mathbb{R}^n$ s.t.

$$c^T x^* < c^T y \quad \text{for all} \quad y \in P,\ y \neq x$$

Extreme point: $x^* \in P$ is extreme point if $\nexists\ y, z \in P$ and $\lambda \in (0,1)$ s.t. $x^* = \lambda y + (1-\lambda) z$

BFS: x^* is BFS if

- $x^* \in P$
- $B = \{ i \mid a_i^T x^* = b_i \}$ with $|B| \geq n$

Extreme point, vertex, BFS.

- For a polyhedron P and $x^* \in P$, TFAE:

  a) x^* is a vertex

  b) x^* is a extreme point

  c) x^* is a BFS.

($c \Rightarrow a$) construct a $c \in \mathbb{R}^n$ s.t. $c^T x < c^T y$ for all $y \in P$.

$c \Rightarrow a$: • Let x^* be a BFS and B be the basic set for x^*.

• Let $c = -\sum_{i \in B} a_i$. Then

$$c^T x^* = -\sum_{i \in B} a_i^T x^* = -\sum_{i \in B} b_i \quad \text{and}$$

$$c^T x = -\sum_{i \in B} a_i^T x \geq -\sum_{i \in B} b_i \quad \text{for any } x \in P.$$

$\Rightarrow$ $x^* = \arg\min \; c^T x$ subject to $x \in P$.

- Also, $-\sum_{i \in B} a_i^T x \geq -\sum_{i \in B} b_i$ holds with equality

  if and only if $a_i^T x = b_i$ for all $i \in B$.

- x^* is unique solution of $a_i^T x = b_i$, $i \in B$.

- So, $c^T x^* < c^T y$ for all $y \in P$.

LP solutions are on extreme points.

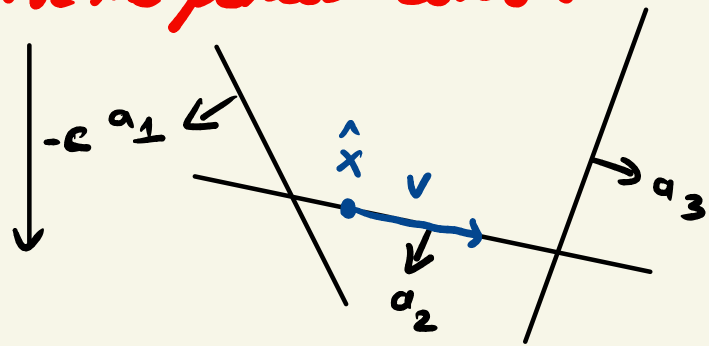
$$\min_{x \in \mathbb{R}^n} \quad c^T x \quad \text{s.t.} \quad Ax \le b. \quad \text{— LP}$$

- Define p^* the optimal value of LP
- Claim: There exists a extreme point x^* of P where $c^T x^* = p^*$. (as long as $p^* > -\infty$)

Proof: • Suppose $c^T \hat{x} = p^*$, but $\hat{x}$ is not an extreme point.
- Then $B = \{ i \mid a_i^T \hat{x} = b_i \}$ satisfies $|B| < n$
- Then A_B has a non-trivial nullspace, Pick $v \in N(A_B)$
- Either $c^T v = 0$ or $c^T v \neq 0$

Proof: LP solutions are on extreme points contd.

Case 1: suppose $c^T v < 0$.



- pick $\tilde{x} = \hat{x} + \alpha v$, $\alpha > 0$

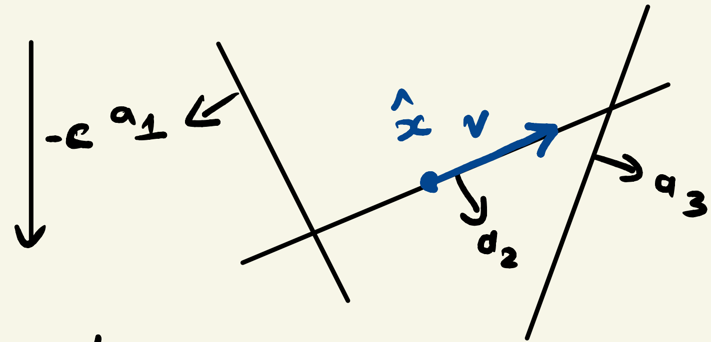
$$\Rightarrow c^T\tilde{x} = c^T\hat{x} + \alpha c^T v$$
$$= p^* + \alpha(-) < p^*$$

- $A_B \tilde{x} = A_B \hat{x}$.
- $A_N \hat{x} < b_N$ $\quad (N = \{1, \dots, m\} \setminus B)$
- Pick α small enough that $A_N \tilde{x} = A_N \hat{x} + \alpha A_N v \leq b_N$

Proof: LP solutions are on extreme points contd.

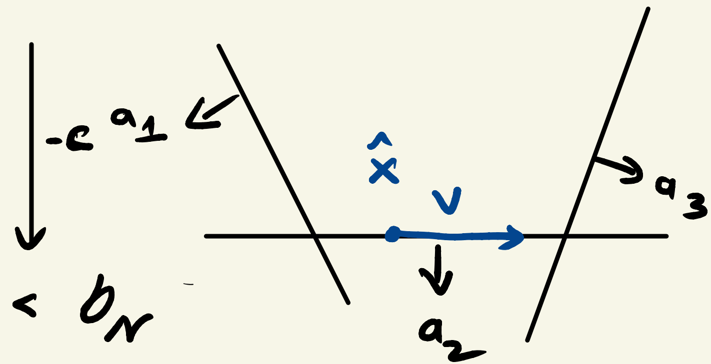
Case 2: Suppose $c^T v > 0$.

- Pick $\tilde{x} = \hat{x} - \alpha v$.
- Then $c^T \tilde{x} = c^T \hat{x} - \alpha c^T v < c^T \hat{x}$
- $A_B \tilde{x} = A_B \hat{x}$
- Pick α small enough that $A_N \tilde{x} < b_N$.



Case 3: Suppose $c^T v = 0$

- Pick $\tilde{x} = \hat{x} + \alpha v$
- Then $c^T \tilde{x} = c^T \hat{x}$ and $A_B \tilde{x} = A_B \hat{x}$
- Pick α small enough that $A_N \tilde{x} < b_N$



$$A_N \tilde{x} = \underbrace{A_N \hat{x}}_{< b_N} + \alpha \underline{A_N v}$$

$b_N + \alpha c_N <$

LP solutions are on extreme points

$$\min_{x \in \mathbb{R}^n} \quad c^T x \quad \text{s.t.} \quad Ax \le b. \quad \text{— LP.}$$

- The optimal cost is $-\infty$ or there exists an optimal solution
- Compare to non-linear function $1/x$, $x \ge 1$
- optimal cost is not $-\infty$, but solution doesnot exist.

Standard Form Polyhedra

Generic polyhedron:

$$P = \left\{ x \;\middle|\; \begin{matrix} Ax = b \\ Cx \le d \end{matrix} \right\} \qquad \begin{matrix} A \in \mathbb{R}^{m \times n} \\ C \in \mathbb{R}^{k \times} \end{matrix}$$

standard form polyhedron

$$P = \left\{ x \;\middle|\; \begin{matrix} Ax = b \\ x \ge 0 \end{matrix} \right\}, \quad b \ge 0$$

Converting to standard form: positive b

- Elements of both b and d (in generic form) will be b in standard form.

For $b_i < 0$, replace

$$a_i^T x = b_i \longrightarrow (-a_i)^T x = (-b_i)$$

For $d_i < 0$, replace

$$c_i^T x \le d_i \longrightarrow (-c_i)^T x \ge (-d_i)$$

$$c_i^T x \ge d_i \longrightarrow (-c_i)^T x \le (-d_i)$$

≥ 0

Converting to standard form: free variable

- x_i is called a free variable if it has no constraint
  $Ax = b$, $Cx \leq d$, x_i may not be constrained
- There are no free variables in standard form
  - every variable must be non-negative.

Converting free variable

- every free variable x_i is replaced with two new variables x_i' and x_i'' with

$$x_i = x_i' - x_i'' \quad , \quad x_i' \geq 0 \, , \quad x_i'' \geq 0$$

- x_i' encodes positive part of x_i
- x_i'' encodes negative part of x_i

Converting to standard form: slack and surplus.

For every inequality constraint of the form

$$c_i^T x \le d_i \qquad (c_i^T x \ge d_i)$$

introduce a new slack (or surplus) variable s_i,

replacing the inequality with two constraint:

$$c_i^T x + s_i = d_i \qquad \left(\begin{array}{l} c_i^T x - s_i = d_i \\ s_i \ge 0 \end{array} \right)$$

$$s_i \ge 0$$