Standard form

- Standard form
- converting to standard form
- degeneracy in standard for

Standard Form Polyhedra

Generic polyhedron:

$$P = \left\{ x \;\middle|\; \begin{array}{l} Ax = b \\ Cx \le d \end{array} \right\} \qquad \begin{array}{l} A \in \mathbb{R}^{m\times n} \\ C \in \mathbb{R}^{k\times} \end{array}$$

standard form polyhedron

$$P = \left\{ x \;\middle|\; \begin{array}{l} Ax = b \\ x \ge 0 \end{array} \right\}, \quad b \ge 0$$

Converting to standard form: positive b

- Elements of both b and d (in generic form) will be b in standard form.

For $b_i < 0$, replace

$$a_i^T x = b_i \longrightarrow (-a_i)^T x = (-b_i)$$

For $d_i < 0$, replace

$$c_i^T x \le d_i \longrightarrow (-c_i)^T x \ge (-d_i)$$

$$c_i^T x \ge d_i \longrightarrow (-c_i)^T x \le (-d_i)$$

≥ 0

Converting to standard form: free variable

- x_i is called a free variable if it has no constraint

  $Ax = b$, $Cx \le d$, x_i may not be constrained

- There are no free variables in standard form
  - every variable must be non-negative.

Converting free variable

- every free variable x_i is replaced with two new variables x_i' and x_i'' with

$$x_i = x_i' - x_i'', \quad x_i' \ge 0, \quad x_i'' \ge 0$$

- x_i' encodes positive part of x_i
- x_i'' encodes negative part of x_i

Converting to standard form: slack and surplus

For every inequality constraint of the form

$$c_i^T x \le d_i \qquad (c_i^T x \ge d_i)$$

introduce a new slack (or surplus) variable s_i, replacing the inequality with two constraint:

$$c_i^T x + s_i = d_i \qquad \left(c_i^T x - s_i = d_i \right)$$

$$s_i \ge 0 \qquad \left(s_i \ge 0 \right)$$

Basic solution in standard form

x^* is a basic solution if the vectors

$$a_{i_1}, \dots, a_{i_n}, \quad i_j \in B$$

are linearly independent

In standard form, there are

- n variables $(x_1, \dots, x_n)$
- $n+m$ total constraints
  - m equality constraint $(Ax=b)$
  - n inequality constraint $(x \geq 0)$

for any basic solution x,

- B must contain n elements.
- Thus $n-m$ of the inequality constraint

Basic solution in standard form

Choose $n-m$ inequality constraint to be active is the same as choosing $n-m$ variables x_j to be zero. Making x_i zero eliminates i^{th} col^n of A.

This is equivalent to choosing m columns of A! To be a basic solution, we need these columns to be linearly independent. So, permute variables and partition

$$AP = [B \quad N] \quad , \qquad \text{where } B \text{ is non-singular}$$

Now,

$$\bar{A}x = \begin{bmatrix} B & N \\ & I \end{bmatrix} \begin{bmatrix} x_B \\ x_N \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

$$x_N = 0, \quad B x_B = b$$

Full rank assumption

- polyhedron in standard form

$$P = \{x \mid Ax = b,\ x \geq 0\},\quad A \in \mathbb{R}^{m \times n}$$

- Assume $\operatorname{rank}(A) = k < m$ and rows $\{a_{i_1}^T, \dots, a_{i_k}^T\}$ of A are linearly independent.
- Let $Q = \{x \mid a_{i_j}^T x = b_{i_j},\ j = 1, \dots, k,\ x \geq 0\}$
- Then $P = Q$. That is we can assume rows of A are linearly independent without loss of generality.

Degeneracy: inequality form

polyhedron in inequality form:

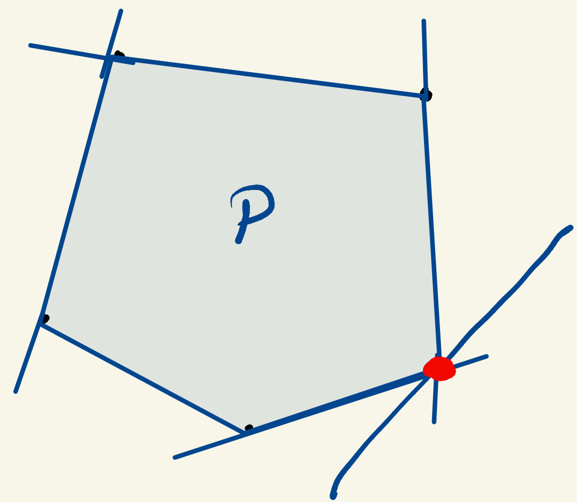
$$Ax \le b$$

a basic feasible solution x^* with

$$a_i^T x^* = b_i, \quad i \in B \quad \text{and} \quad a_i^T x^* < b_i, \quad i \notin B$$

is degenerate if # of indices in B is greater than n.

- property of description of polyhedron
- affects performance of some algorithm
- disappears for small perturbations of b.



Degeneracy: standard form

polyhedron in standard form:

$$Ax = b, \quad x \geq 0$$

a basic solution partitions the variables into two sets:

$$[B, \ N] = b \qquad \text{with} \qquad x_N = 0$$

i.e.

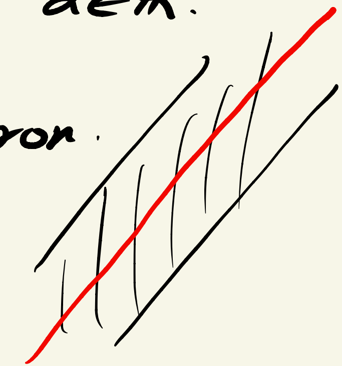
$$Bx_B = b$$

a basic feasible solution in standard form is degenerate if more that $n-m$ components in x are zero, i.e.

$$x = \begin{array}{c} \begin{bmatrix} x_B \\ x_N \end{bmatrix} \end{array} \begin{array}{l} m \\ n-m \end{array} = \begin{bmatrix} B^{-1}b \\ 0 \end{bmatrix} \longleftarrow \text{has some zero components.}$$

Existence of extreme point:

- A polyhedron contains a line if there exists $x \in P$, $d \in \mathbb{R}^n$ such that $x + \alpha d \in P$ for all $\alpha \in \mathbb{R}$.



- Let $P = \{x \mid Ax \le b\}$ be a non-empty polyhedron. $A \in \mathbb{R}^{m \times n}$

TFAE:

(a) The polyhedron has at least one extreme point

(b) The polyhedron doesnot contain a line

(c) There exists n rows of A linearly independent

- Every bounded, non-empty polyhedron has an extreme point.
- Polyhedron is standard form always has an extreme point.