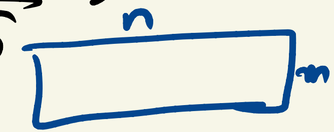
Simplex method

- computing BFS
- computing feasible direction
- reduced cost

Assumptions

Develop simplex method for a LP in standard form:

$$\min_{x} \; c^T x \qquad \text{subject to} \qquad \underline{Ax = b}, \quad A \in \mathbb{R}^{m \times n}$$
$$x \geq 0$$



we will assume:

- A has full row rank (no redundant rows)
- LP is feasible (need to check if unbounded).
- all <u>basic feasible solutions</u> are non degenerate.

  → extreme point, vertex. $\quad x = \begin{bmatrix} x_B \\ x_N \end{bmatrix}$

variable index set: ← indices of columns of A.

$\mathcal{B} = \{\beta_1, \dots, \beta_m\}$ ← basic variable $\qquad A = [B \;\; N]$

$\mathcal{N} = \{\eta_1, \dots, \eta_{n-m}\}$ ← non-basic variable.

Constructing basic feasible solution

Basic feasible solution partitions the columns of A:

$$AP = [\underbrace{B}_{\in \mathbb{R}^{m\times m}} \quad N] \in \mathbb{R}^{m\times n}, \quad \text{where } B \text{ is non-singular}$$

$$x_B = B^{-1}b, \quad x_N = 0$$

$$Ax = b$$

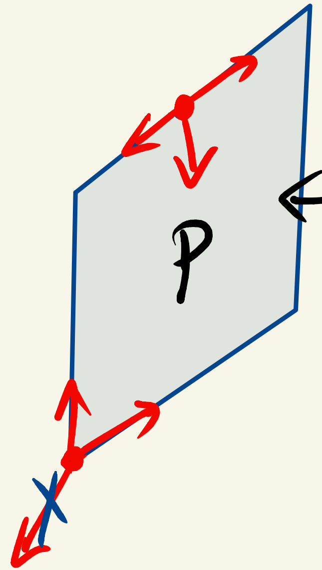
$$[B \quad N]\begin{bmatrix} x_B \\ x_N \end{bmatrix} = b.$$

Procedure for constructing BFS:

1. Choose m linearly independent columns of A. $\quad A = [B \quad N]$
2. $x_N = 0 \quad (x_i = 0 \text{ for } i \notin B)$
3. $x_B = B^{-1}b \in \mathbb{R}^m$
4. check if $x_B \geq 0$.

Feasible direction.

A direction $d \in \mathbb{R}^n$ is a feasible direction at $x \in P$ if $x + \alpha d \in P$ for some $\alpha > 0$



$Ax = b$
$x \geq 0$

Constructing a feasible direction

given $x \in P$ with $Ax = b$, $x \geq 0$

require for $\alpha > 0$ that

$$b = A(x + \alpha d) = b + \alpha Ad$$

Thus, we require $Ad = 0$ $\leftarrow$ d is in the nullspace of A.

$A = [B \;\; N]$

Suppose x is BFS so that

$$Ad = [B \;\; N] \begin{bmatrix} d_B \\ d_N \end{bmatrix} = Bd_B + Nd_N$$

$$\Rightarrow \quad Bd_B = -Nd_N \longrightarrow \quad d_N = e_k$$

construct search direction by moving a single variable $\eta_k \in \mathcal{N}$:

$$Bd_B = -N\underbrace{e_k}_{\in \mathbb{R}^{n-m}} = -a_{\eta_k}$$

select a column form non-basic $\mathcal{N}$

$d = \begin{bmatrix} d_B \\ d_N \end{bmatrix}$

Example

$\boxed{\begin{array}{l} Ax = b \\ x \geq 0 \end{array}}$ $\quad x_j + \alpha d_j = 0$

$d_j = -\dfrac{x_j}{d_j}$

minimize $\quad c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4$

subject to

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 2 \\ 2x_1 + \quad\quad + 3x_3 + 4x_4 &= 2 \\ x &\geq 0 \end{aligned}$$

$x_j > 0$

$x_j = 0$

$\min \left\{ \dfrac{-x_j}{d_j} \,\middle|\, j \in B \right\}$

$x_N = 0$

$B \rightarrow$

$x_N + \alpha d_N \geq 0$

Construct basic feasible sol^n.

$$\underbrace{B = \{1, 2\}}_{} \Rightarrow B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} x_B = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \Rightarrow x_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow x = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$N = \{3, 4\} \quad x_N = 0$

increase a non-basic variable, say x_3. That is, $d_N = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$B d_B = -N d_N \Rightarrow \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} d_B = -\begin{bmatrix} 1 \\ 3 \end{bmatrix} \Rightarrow d_B = \begin{bmatrix} 1/2 \\ -3/2 \end{bmatrix}$$

$$d = (1/2, -3/2, 1, 0) \Rightarrow Ad = 0,\ A(x + \alpha d) = b,\ x + \alpha d \geq 0 \ ?$$

Change in objective

objective value at $\bar{x} = x + \alpha d$, where d is feasible direction.

how does $c^T \bar{x} = c^T(x + \alpha d)$ change?

Let $\phi(x) = \phi = c^T x$.

$$\begin{aligned}
\bar{\phi} = \phi(\bar{x}) &= c^T(x + \alpha d) \\
&= c^T x + \alpha \begin{bmatrix} c_B^T & c_N^T \end{bmatrix} \begin{bmatrix} d_B \\ d_N \end{bmatrix} \\
&= \phi + \alpha \left(c_B^T d_B + c_N^T d_N \right). \\
&= \phi + \alpha \underbrace{\left(c_{q_k} + c_B^T d_B \right)}_{\text{reduced cost.}}.
\end{aligned}$$

$$\begin{aligned}
B d_B &= -N d_N \\
&= -a_{q_k}
\end{aligned}$$

if $d_N = e_k$, k^{th} non-basic variable increases.

Reduced cost

$\begin{bmatrix} x_B \\ x_N \end{bmatrix} \geq 0$

reduce cost for any variable $j = 1, \dots, n$.

$$z_j := c_j - c_B^T B^{-1} a_j$$

if $z_j \geq 0$ for all j

reduced cost for basic variable, $j \in B$.

$$z_j := c_j - c_B^T B^{-1} a_j = c_j - c_B^T B^{-1} B e_j$$

$$= 0$$

thus only non-basic variables need to be considered.

note if $z \geq 0$, then all feasible directions increase objective value.

__Thm:__ consider a basic feasible solution x with reduced cost vector z. $z_j < 0$

- If $z \geq 0$ then x is optimal.
- If x is optimal and non-degenerate, then $z \geq 0$.