Simplex method

- step length
- Blocking variable & basis change
- optimality

$$B d_B = -N d_N$$

$$\phi(\bar{x}) = \phi(x) + \alpha c^T d$$

$$= \phi + \alpha\left(c_B^T d_B + c_N^T d_N\right)$$

$$= \phi + \alpha\left(\underbrace{c_B^T d_B + c_{n_p}}_{z_{n_p}}\right)$$

$$z = \begin{bmatrix} \; \end{bmatrix} \in \mathbb{R}^n \quad n_p$$

Reduced cost

$$\boxed{\phi(\bar{x}) = \phi(x) + \alpha \cdot z_j}$$

reduced cost for any variable x_j, $j = 1, \dots, n$:

$$z_j := c_j + c_B^T d_B = c_j - c_B^T B^{-1} a_j$$

$$\bar{x} = x + \alpha d$$

reduced cost for a basic variable, $\boxed{j \in B}$

$$z_j = c_j - c_B^T B^{-1} a_j$$
$$= c_j - c_j$$
$$= 0$$

$$z = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \geq 0$$

$$\phi(\bar{x}) = \phi(x) + \alpha \cdot z_j$$

thus only non-basic variables need to be considered.

note: if $z \geq 0$, then all feasible directions increase objective.

<u>Thm:</u> consider a basic feasible solution x with reduced cost z.

- if $z \geq 0$ then x is optimal
- if x is optimal and non-degenerate then $z \geq 0$.

Choosing a stepsize

change in objective value from moving p^{th} non-basic variable. $\eta_p \in N$ $(B = \{\beta_1, \dots \beta_m\}, \; N = \{\eta_1, \dots, \eta_{n-m}\})$

$B d_B = -N d_N$

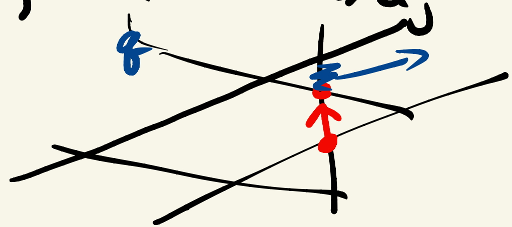
$$\phi(\bar{x}) = \phi + \alpha z_{\eta_p}, \quad z_j = c_j - c_B^T B^{-1} a_j$$

$z_{\eta_p} < 0$, so choose α as large as possible. $(Ax = b)$ $\boxed{x + \alpha d \geq 0}$

$d = \begin{bmatrix} d_B \\ d_N \end{bmatrix}$ $\quad \alpha^* = \max\{\alpha \geq 0 \mid x + \alpha d \geq 0\}$.

Case 1: if $d \geq 0$ then, it is unbounded feasible direction of descent i.e. $x + \alpha d \geq 0$ for all $\alpha \geq 0$.

case 2: if $d_j < 0$, then $x_j + \alpha d_j \geq 0$ if $\alpha \leq -x_j/d_j$ for every $d_j < 0$.



$\Rightarrow$ ratio test: $\alpha^* = \min\{-\frac{x_j}{d_j} \mid j \in B, d_j < 0\}$

Basis change

Case 1: no "blocking" variable. Therefore d is a direction of unbounded descent

Case 2 the first basic variable to "hit" bound is "blocking"!

$$\boxed{\bar{B} x_B = b}$$

Variable swap:

- enters basic variable $\eta_p \in N$ becomes basic
  ($\bar{x}_{\eta_p} \to +$) (before $x_{\eta_p} = 0$)
- blocking variable $\beta_q \in B$ becomes non-basic.
  ($\bar{x}_{\beta_q} = 0$)

$$A = \left[\, B \mid N \,\right]$$

- $B \leftarrow (B \setminus \{\beta_q\}) \cup \{\eta_p\}$, $N \leftarrow (N \setminus \{\eta_p\}) \cup \{\beta_q\}$.

A new basis

$\beta_q \leftarrow \eta_p$

the new set of columns define a basic feasible solution.

The new basis matrix:

$$\bar{B} = [a_{\beta_1} \dots a_{\eta_p} \dots a_{\beta_m}] \quad \text{with } \eta_p \in N.$$

has rank m. Note:

$$B^{-1}B = B^{-1}[a_{\beta_1} \dots a_{\beta_q} \dots a_{\beta_m}] = B^{-1}[Be_1 \dots Be_m]$$
$$= [e_1 \dots e_m] = I.$$

$$B^{-1}\bar{B} = B^{-1}[a_{\beta_1} \dots a_{\eta_p} \dots a_{\beta_m}]$$
$$= [e_1 \dots B^{-1}a_{\eta_p} \dots e_m].$$

$Bd_B = -Nd_N$

$$= [e_1 \dots -d_B \dots e_m].$$

$$= \begin{bmatrix} 1 & 0 & & -d_{\beta_1} & & \\ 0 & 1 & & \vdots & & \\ \vdots & & 0 & \cdots -d_{\beta_p} & & \\ \vdots & & \vdots & \vdots & \ddots & \\ 0 & 0 & & -d_{\beta_m} & & 1 \end{bmatrix}$$

matrix is invertible.

invertible because $d_{\beta_p} < 0$

Simplex without B^{-1}

search direction: maintain $Ax = b$ and $A(x+\alpha d) = b$ for all $\alpha \geq 0$.

$$Ad = [B \;\; N]\begin{bmatrix} d_B \\ d_N \end{bmatrix} \Rightarrow B d_B = -N d_N = -a_{n_p}$$

effect on objective: $(z_j = c_j - c_B^T B^{-1} a_j)$

$$B^T y = c_B \Rightarrow y = B^{-T} c_B = (c_B^T B^{-1})^T, \quad \underline{z = c - A^T y} = \begin{bmatrix} z_B \\ z_N \end{bmatrix}$$

↳ simplex "multiplier".

$$\phi(\bar{x}) = \phi(x) + \alpha c^T d = \phi(x) + \alpha (c_B^T d_B + c_N^T d_N)$$

$$= \phi + \alpha (y^T B d_B + c_N^T d_N)$$

$$= \phi + \alpha (-y^T N d_N + c_N^T d_N)$$

$$= \phi + \alpha (c_N - N^T y)^T d_N = \phi + \alpha \cdot z_N^T d_N$$

$$c_B^T B^{-1} a_j = (A^T y)_j$$

choose p so that $z_{\eta_p} < 0$ (eg: most negative). non-basic x_p enters basic.

<u>optimality</u>: no improving direction exist if for each $j = 1, \dots, n$

$$x_j \cdot z_j = 0 \qquad\qquad x_j = 0 \text{ and } z_j \geq 0$$

$$x_j \geq 0,\ z_j \geq 0 \qquad\qquad x_j \geq 0 \text{ and } z_j = 0$$

<u>ratio test</u>: basic variable β_q exists basic

$$q = \operatorname*{arg\,min}_{q \,/\, d_q < 0} \frac{-x_{\beta_q}}{d_{\beta_q}}$$

Example

$$A = \begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix}, \quad c = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

iteration 0: $\mathcal{B} = \{3, 4, 5\}$, $\mathcal{N} = \{1, 2\}$

Which of the following is the new basic set (and non-basic set)?

1. $\mathcal{B} = \{2, 4, 5\}$, $\mathcal{N} = \{1, 3\}$.
2. $\mathcal{B} = \{3, 2, 5\}$, $\mathcal{N} = \{1, 4\}$
3. $\mathcal{B} = \{1, 4, 5\}$, $\mathcal{N} = \{2, 3\}$

Example

$$A = \begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix}, \quad c = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

iteration 0: $\mathcal{B} = \{3, 4, 5\}$, $\mathcal{N} = \{1, 2\}$

Example contd

$$\boxed{\begin{array}{c} x_j \cdot z_j \geq 0 \\ z \geq 0,\ x \geq 0 \end{array}}$$

iteration 2:

- current basis: $\mathcal{B} = \{2,4,5\}$, $\mathcal{N} = \{1,3\}$
- $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- solve $B x_B = b \rightarrow x_B = (2,3,3)$ $\quad [x_N = 0]$
- simplex multiplier: $B^T y = c_B \longrightarrow y = (-2,0,0)$
- reduced cost: $z_N = c_N - N^T y \rightarrow z_N = (-5,2)$, $[z_B = 0]$
- choose $\eta_1 = 1$ to enter basis.
- search direction: $B d_B = -a_1 \rightarrow d_B = (2,-3,1)$
- ratio test: $q = \underset{q \mid d_{\beta_q} < 0}{\arg\min} \dfrac{-x_{\beta_q}}{d_{\beta_q}} \longrightarrow q = 2,\ \beta_q = 4$ exits
- $\mathcal{B} = \{2,1,5\}$, $\mathcal{N} = \{4,3\}$

iteration 3:

- current basis $\mathcal{B} = \{2, 1, 5\}$, $\mathcal{N} = \{4, 3\}$
- $B = \begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, $B^{-1} = (1/3) \begin{bmatrix} -1 & 2 & 0 \\ -2 & 1 & 0 \\ 2 & -1 & 3 \end{bmatrix}$
- solve : $Bx_B = b \rightarrow x_B = (4, 1, 2)$
- simplex multiplier : $B^T y = c_B \rightarrow y = (4/3, -5/3, 0)$
- reduced cost : $z_N = c_N - N^T y \rightarrow z_N = (5/3, -4/3)$
- choose $\eta_2 = 3$ to enter basis
- search direction : $B d_B = -a_3 \rightarrow d_B = (1/3, 2/3, -2/3)$
- ratio test : $q = -3$, $\beta_q = 5$ exits basic
- $\mathcal{B} = \{2, 1, 3\}$, $\mathcal{N} = \{4, 5\}$.

iteration 4:

- current basis: $\mathcal{B} = \{2, 1, 3\}$, $\mathcal{N} = \{4, 5\}$

- $B = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \\ 1 & -1/2 & 3/2 \end{bmatrix}$

- solve $Bx_B = b \rightarrow x_B = (4, 1, 2)$

- simplex multiplier: $B^T y = c_B \longrightarrow y = (0, -1, -2)$

- reduced cost: $z_N = c_N - N^T y \rightarrow z_N = (1, 2)$

- $z_N \geq 0 \longrightarrow$ basis is optimal.