Simplex method

- Example
- General bound
- 2-phase simplex

Example

$$\min \ c^T x \quad \text{s.t.} \quad Ax = b, \quad x \geq 0$$

$$A = \begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix}, \quad c = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

iteration 0: $\mathcal{B} = \{3, 4, 5\}$, $\mathcal{N} = \{1, 2\}$

Which of the following is the new basic set (and non-basic set)?

1. $\mathcal{B} = \{2, 4, 5\}$, $\mathcal{N} = \{1, 3\}$.
2. $\mathcal{B} = \{3, 2, 5\}$, $\mathcal{N} = \{1, 4\}$
3. $\mathcal{B} = \{1, 4, 5\}$, $\mathcal{N} = \{2, 3\}$

Example

$Ad = 0$

$$A = \begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix}, \quad c = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

iteration 0: $\mathcal{B} = \{3, 4, 5\}$, $\mathcal{N} = \{1, 2\}$

iteration 1:

- $B = I$, $[N \ \ B] \begin{bmatrix} x_N \\ x_B \end{bmatrix} = b$
- solve $B x_B = b \Rightarrow x_B = (2, 7, 3) \geq 0$ ok.
- simplex multiplier: $B^T y = c_B \rightarrow y = 0$
- reduced cost: $z_N = c_N - N^T y \rightarrow z_N = c_N = (-1, -2)$
- choose n_2 to enter basic set.
- search direction: $B d_B = -a_2 \rightarrow d_B = (-1, -2, 0)$
- ratio test: $q = \underset{q / d_{\beta_q} < 0}{\arg\min} \dfrac{-x_{\beta_q}}{d_{\beta_q}} \rightarrow q = 1$, $\beta_q = 3$

exit

new iteration: $\mathcal{B} = \{2, 4, 5\}$, $\mathcal{N} = \{1, 3\}$

Example contd

iteration 2:

- current basis: $\mathcal{B} = \{2,4,5\}$, $\mathcal{N} = \{1,3\}$

$$\boxed{\begin{array}{c} x_j \cdot z_j \geq 0 \\ z \geq 0,\ x \geq 0 \end{array}}$$

- $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- solve $B x_B = b \rightarrow x_B = (2,3,3)$ $[x_N = 0]$
- simplex multiplier: $B^T y = c_B \longrightarrow y = (-2,0,0)$
- reduced cost: $z_N = c_N - N^T y \rightarrow z_N = (-5,2)$, $[z_B = 0]$
- choose $\eta_1 = 1$ to enter basis.
- search direction: $B d_B = -a_1 \rightarrow d_B = (2,-3,1)$
- ratio test: $q = \underset{q \mid d_{\beta_q} < 0}{\arg\min} \dfrac{-x_{\beta_q}}{d_{\beta_q}} \longrightarrow q = 2$, $\beta_q = 4$ exits
- $\mathcal{B} = \{2,1,5\}$, $\mathcal{N} = \{4,3\}$

iteration 3:

- current basis $\mathcal{B} = \{2, 1, 5\}$, $\mathcal{N} = \{4, 3\}$
- $B = \begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, $B^{-1} = (1/3) \begin{bmatrix} -1 & 2 & 0 \\ -2 & 1 & 0 \\ 2 & -1 & 3 \end{bmatrix}$
- solve: $Bx_B = b \rightarrow x_B = (4, 1, 2)$
- simplex multiplier: $B^T y = c_B \rightarrow y = (4/3, -5/3, 0)$
- reduced cost: $z_N = c_N - N^T y \rightarrow z_N = (5/3, -4/3)$
- choose $\eta_2 = 3$ to enter basis
- search direction: $B d_B = -a_3 \rightarrow d_B = (1/3, 2/3, -2/3)$
- ratio test: $q = -3$, $\beta_q = 5$ exits basic
- $\mathcal{B} = \{2, 1, 3\}$, $\mathcal{N} = \{4, 5\}$.

iteration 4:

- current basis : $\mathcal{B} = \{2, 1, 3\}$, $\mathcal{N} = \{4, 5\}$
- $B = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \\ 1 & -1/2 & 3/2 \end{bmatrix}$
- solve $Bx_B = b \rightarrow x_B = (4, 1, 2)$
- simplex multiplier : $B^T y = c_B \longrightarrow y = (0, -1, -2)$
- reduced cost : $z_N = c_N - N^T y \rightarrow z_N = (1, 2)$
- $z_N \geq 0 \longrightarrow$ basis is optimal.

$$B^T y = c_B$$

$$z_j := c_j - c_B^T B^{-1} a_j$$

$$\phi(\bar{x}) = \phi + \alpha(c_B^T d_B + c_N^T d_N)$$

$$\bar{\phi} = \phi + \alpha \, z_N^T d_N$$

General upper and lower bound

standard form

$$\min_{x} \quad c^T x$$

$$\text{s.t.} \quad Ax = b \ , \ x \geq 0$$

$x_N = 0$

general bound

$$l \leq x \leq u$$

reduction to standard form

$$\begin{cases} x - s_1 = l \ , & s_1 \geq 0 \quad \Rightarrow \quad l \leq x \\ x + s_2 = u \ , & s_2 \geq 0 \quad \Longrightarrow \quad x \geq u \end{cases}$$

standard form problem

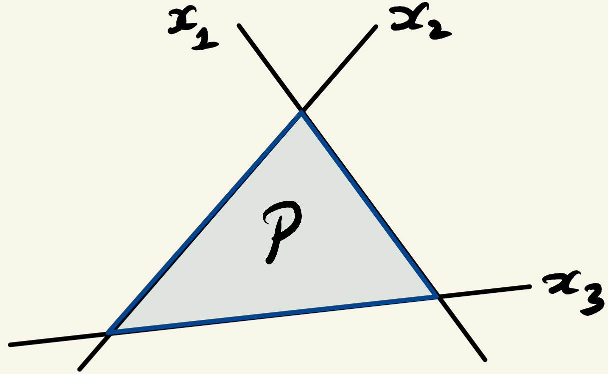
$A \in \mathbb{R}^{m \times n}$

$$\min_{x, s_1, s_2} \quad c^T x$$

$$\text{subject to} \quad \begin{bmatrix} A & & \\ I & -I & \\ I & & I \end{bmatrix} \begin{bmatrix} x \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} b \\ l \\ u \end{bmatrix}$$

$$\bar{A} \in \mathbb{R}^{2m \times (n+2m)} \quad , \quad \begin{bmatrix} x \\ s_1 \\ s_2 \end{bmatrix} \geq 0$$

General bounds and non basic variable



- nonbasic variable are always at their bounds
- basic variables are uniquely determined by non-basic variables.

$$b = Ax = [B \quad N]\begin{bmatrix} x_B \\ x_N \end{bmatrix} = Bx_B + Nx_N$$

thus,

$$Bx_B = b - Nx_N \,, \quad (x_N)_j = l_j \quad \text{or} \quad (x_N) = u_j$$

$$j = \eta_1, \dots, \eta_{n-m}$$

Simplex with general bounds

effect on objective: need to choose a "good" d_N. Solve

$$B^T y = c_B \quad \text{and} \quad z := c - A^T y$$

$z_N = 0$

$x \geq 0$

$$\bar{\phi} = \phi + \alpha z_N^T d_N$$

pricing: only one non-basic variable x_p moves, implying

$$d_N = \pm e_p \quad , \quad B d_B = \pm a_{x_p} \quad (-N d_N)$$

so choose p so that,

$$l_{x_p} = x_{x_p} \quad \text{and} \quad z_{x_p} < 0 \Rightarrow \text{set } d_{x_p} = 1$$

$$u_{x_p} = x_{x_p} \quad \text{and} \quad z_{x_p} > 0 \Rightarrow \text{set } d_{x_p} = -1$$

optimality: no improving direction exists if for $j = 1, \dots, n$

$$l_j = x_j \quad \text{and} \quad z_j \geq 0$$

$$x_j = u_j \quad \text{and} \quad z_j \leq 0$$

$$l_j \leq x_j \leq u_j \quad \text{and} \quad z_j = 0$$

Finding an initial Basic Feasible Solution

not always obvious how to choose a linearly independent set of columns of B so that $x \geq 0$

$$[B\ N]\begin{bmatrix} x_B \\ x_N \end{bmatrix} = b, \quad Bx_B = b - \cancel{N x_N}^{0} = b$$

need $x_B \geq 0$ for $\begin{bmatrix} x_B \\ x_N \end{bmatrix}$ to be BFS.

phase 1: (auxiliary LP)

$$\min_{x,s} \ e^T s$$

subject to

$$Ax + s = b \quad \rightarrow \quad [A\ I]\begin{bmatrix} x \\ s \end{bmatrix} = b$$

$$\begin{bmatrix} x \\ s \end{bmatrix} \geq 0.$$

original:

$$\min_x \ c^T x \quad (1)$$

subject to

$$Ax = b$$
$$x \geq 0$$

where s is an "artificial" variable.

If $s^* = 0$ then x^* is BFS of (1)

Two Phase Simplex

Phase 1:

- ensure $b \geq 0$
- $\bar{A} = [A \ \ I]$, $\bar{x} = (x, s)$, $\bar{x} \geq 0$. $B = \{n+1, \ldots, n+m\}$

  $$[A \ \ I] \begin{bmatrix} x_N \\ x_B \end{bmatrix} = b \Rightarrow x_B = b$$

  Apply simplex to auxilary program.
- If solution (x^*, s^*) has $s^* \neq 0$, original LP is not feasible.
- If $s^* = 0$, then original LP is feasible.
- optimal basis will have to auxilary variables because non-degeneracy assumption.

Phase 2: 1. use optimal basis from Phase 1 for BFS
2. simplex method for LP.