Duality

- LP dual
- weak duality
- strong duality
- complementarity

Duality.

Consider the constrained optimization problem

$$\min_{x_1, x_2} \quad x_1^2 + x_2^2 \qquad \text{subject to} \quad x_1 + x_2 = 1$$

and the unconstrained problem:

$$\min_{x_1, x_2} \quad \phi(x_1, x_2, y) := x_1^2 + x_2^2 + y(1 - x_1 - x_2)$$

$\phi(x_1, x_2, y)$ is the Lagrangean and the scalar y is the "price" for violating the constraint $x_1 + x_2 = 1$.

$$\left.\begin{aligned} \frac{\partial \phi}{\partial x_1} &= 2x_1 - y = 0 \\ \frac{\partial \phi}{\partial x_2} &= 2x_2 - y = 0 \end{aligned}\right\} \Rightarrow x_1 = x_2 = \frac{y}{2} \xRightarrow{x_1 + x_2 = 1} y^* = 1$$

$y^* = 1$ induces the optimal solution $x^* = (\frac{1}{2}, \frac{1}{2})$.

Dual Function: LP

primal problem: $\min_x \; c^T x$ subject to $Ax = b, \; x \geq 0$

- n variables & m constraints.
- optimal value is p^* with x^* as optimal variable

relaxed problem: $\min_x \; c^T x + y^T(b - Ax), \; x \geq 0$.

- replacing constraint $Ax = b$ by penalty $y^T(b - Ax)$.
- relaxed problem is a lower bound for p^*.

$$g(y) := \min_{x \geq 0} \{ c^T x + y^T(b - Ax) \} \leq c^T x^* + y^T(b - Ax^*)$$
$$= c^T x^* = p^*$$

tightest lower bound: find y such that

$$\max_y \; g(y)$$ — no constraint (closest to p^*)

Main result: optimal cost of dual program equals optimal primal cost.

Dual of an LP

using definition of $g(y)$:

$$g(y) = \min_{x \geq 0} \{ c^T x + y^T (b - Ax) \}$$

$$= b^T y + \min_{x \geq 0} \{ x^T (c - A^T y) \}$$

$$= \begin{cases} b^T y & \text{if } c - A^T y \geq 0 \\ -\infty & \text{otherwise.} \end{cases}$$

Because we want to maximize $g(y)$, we only need to consider values of y such that $g(y) \neq -\infty$.

$$\begin{aligned} \max_{y} \quad & b^T y \\ \text{subject to} \quad & c - A^T y \geq 0 \end{aligned} \iff \begin{aligned} \max_{y,z} \quad & b^T y \\ \text{subject to} \quad & A^T y + z = c. \\ & z \geq 0 \end{aligned}$$

This is dual LP.

Weak duality

Suppose x is primal feasible vector:

$$Ax = b, \quad x \geq 0$$

Suppose (y, z) is dual feasible vector.

$$A^T y + z = c \quad , \quad z \geq 0$$

Then,

$$c^T x = (A^T y + z)^T x = y^T A x + z^T x = y^T b + z^T x \geq y^T b.$$

$c^T x$ is an upper bound for $y^T b$ for any x, (y, z).

<u>Weak duality theorem</u>: If (x, y, z) is primal/dual feasible, then:

- The primal value is an upper bound for dual value

Complementarity

primal

$$\min_x \; c^T x$$

subject to $Ax = b$

$x \geq 0$

dual

$$\max_y \; b^T y$$

subject to $A^T y + z = c$

$z \geq 0$

primal value $\equiv c^T x = b^T y + z^T x \geq b^T y \equiv$ dual value.

The bound is "tight" when x and z are complementary:

$z^T x = 0$

$x_j = 0$ and $z_j \geq 0$

$x_j \geq 0$ and $z_j = 0$

Optimality condition.

Simplex method maintains primal feasibility at every iteration: $Ax=b, \quad x\ge 0$

It defines y via $B^Ty=c_B$ and $z=c-A^Ty$ and maintain complimentarity:

non-degeneracy → $\boxed{x_B\ge 0}$ and $z_B=0$ (by construction)

$[B\ N]\begin{bmatrix}x_B\\ \cancel{x_N}\end{bmatrix}=b$ $\quad 0$

$x_N=0$ and $z_N \lesseqgtr 0$

Simplex exists when $z_N\ge 0$, i.e. (y,z) is dual feasible. $\quad A^Ty+z=c, \quad z\ge 0$

$$\boxed{\begin{aligned} y &= B^{-T}c_B \text{ and} \\ b^Ty &= \underline{b^TB^{-T}c_B} \end{aligned}}$$

$$= x_B^Tc_B = p^*$$

<u>Strong duality theorem:</u> If LP has an optimal solution, so does its dual and the optimal values are equal i.e. $p^*=d^*$

$\phi(x) = c^T x$, $\phi(x+\alpha d) = \phi + \alpha c^T d$

$= \phi + \alpha(c_B^T d_B + c_N^T d_N)$

$Ad = 0 \Rightarrow \underbrace{Bd_B = -Nd_N}$ $= \phi + \alpha(-c_B^T B^{-1} N d_N + c_N d_N)$

$\Rightarrow d_B = B^{-1} N d_N$ $= \phi + \alpha(-\underbrace{c_B^T B^{-1}} a_{n_j} + c_{n_j})$

$= \phi + \alpha(\underbrace{-y^T a_{n_j} + c_{n_j}})$

n_j^{th} component of z_{n_j}

$B^T y = c_B$

$y = B^{-T} c_B$

$y = (c_B^T B^{-1})^T$

$z = c - A^T y$

Sufficient condition.

Relationship between primal and dual LPs.

	finite optimal	unbounded	infeasible
finite optimal			
unbounded			
infeasible			

Interpretation of dual variables