

# Duality

- LP dual
- weak duality
- strong duality
- complementarity

# Duality

Consider the constrained optimization problem

$$\min_{x_1, x_2} x_1^2 + x_2^2 \quad \text{subject to } x_1 + x_2 = 1$$

and the unconstrained problem:

$$\min_{x_1, x_2} \phi(x_1, x_2, y) = x_1^2 + x_2^2 + y(1 - x_1 - x_2)$$

$\phi(x_1, x_2, y)$  is the Lagrangean and the scalar  $y$  is the "price" for violating the constraint  $x_1 + x_2 = 1$ .

$$\left. \begin{aligned} \frac{\partial \phi}{\partial x_1} = 2x_1 - y = 0 \\ \frac{\partial \phi}{\partial x_2} = 2x_2 - y = 0 \end{aligned} \right\} \Rightarrow x_1 = x_2 = \frac{y}{2} \Rightarrow y^* = 1$$

$y^* = 1$  induces the optimal solution  $x^* = (\frac{1}{2}, \frac{1}{2})$ .

## Dual function: LP

primal problem:  $\min_x c^T x$  subject to  $Ax = b, x \geq 0$

- $n$  variables &  $m$  constraints.
- optimal value is  $p^*$ . with  $x^*$  as optimal variable

relaxed problem:  $\min_x c^T x + y^T (b - Ax), x \geq 0$ .

- replacing constraint  $Ax = b$  by penalty  $y^T (b - Ax)$ .
- relaxed problem is a lower bound for  $p^*$ .

$$g(y) := \min_{x \geq 0} \{ c^T x + y^T (b - Ax) \} \leq c^T x^* + y^T (b - Ax^*) = c^T x^* = p^*$$

tightest lower bound: find  $y$  such that

$$\max_y g(y) \quad - \text{no constraint (closest to } p^*)$$

Main result: optimal cost of dual program equals optimal primal cost.

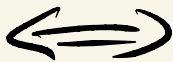
## Dual of an LP

using definition of  $g(y)$ :

$$\begin{aligned} g(y) &= \min_{x \geq 0} \{ c^T x + y^T (b - Ax) \} \\ &= b^T y + \min_{x \geq 0} \{ x^T (c - A^T y) \} \\ &= \begin{cases} b^T y & \text{if } c - A^T y \geq 0 \\ -\infty & \text{otherwise.} \end{cases} \end{aligned}$$

Because we want to maximize  $g(y)$ , we only need to consider values of  $y$  such that  $g(y) \neq -\infty$ .

$$\begin{aligned} &\max_y b^T y \\ &\text{subject to } c - A^T y \geq 0 \end{aligned}$$



$$\begin{aligned} &\max_{y, z} b^T y \\ &\text{subject to } Ay + z = c \\ &z \geq 0 \end{aligned}$$

This is dual LP.

## Weak duality

Suppose  $x$  is primal feasible vector:

$$Ax = b, \quad x \geq 0$$

Suppose  $(y, z)$  is dual feasible vector.

$$A^T y + z = c, \quad z \geq 0$$

$$\text{Then, } c^T x = (A^T y + z)^T x = y^T A x + z^T x = y^T b + z^T x \geq y^T b.$$

$c^T x$  is on upper bound for  $y^T b$  for any  $x, (y, z)$ .

Weak duality theorem: If  $(x, y, z)$  is primal / dual feasible, then:

- The primal value is on upper bound for dual value

# Complementarity

primal

dual

$$\begin{aligned} \min_x \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

$$\begin{aligned} \max_y \quad & b^T y \\ \text{subject to} \quad & A^T y + z = c \\ & z \geq 0 \end{aligned}$$

$$\text{primal value} \equiv c^T x = b^T y + z^T x \geq b^T y \equiv \text{dual value.}$$

The bound is "tight" when  $x$  and  $z$  are complementary:

$$z^T x = 0$$

$$x_j = 0 \quad \text{and} \quad z_j \geq 0$$

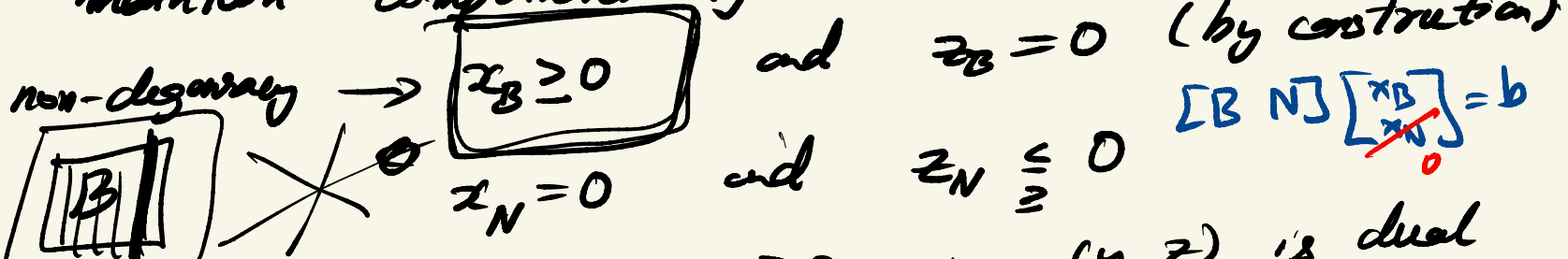
$$x_j \geq 0 \quad \text{and} \quad z_j = 0$$

# Optimal condition.

Simplex method maintains primal feasibility at every iteration:  
 $Ax = b, \quad x \geq 0$

It defines  $y$  via  $B^T y = c_B$  and  $z = c - A^T y$  and

maintain complementarity:



Simplex exists when  $z_N \geq 0$ , i.e.  $(y, z)$  is dual

feasible.  $A^T y + z = c, \quad z \geq 0$

$$y = B^{-T} c_B \text{ and}$$

$$b^T y = b^T B^{-T} c_B$$

$$= x_B^T c_B = p^*$$

Strong duality theorem: If LP has an optimal solution, so does its dual and the optimal values are equal i.e.  $p^* = d^*$

$$\phi(x) = c^T x, \quad \phi(x+\alpha d) = \phi + \alpha c^T d$$

$$= \phi + \alpha (c_B^T d_B + c_N^T d_N)$$

$$Ad=0 \Rightarrow \underbrace{Bd_B = -Nd_N}$$

$$\Rightarrow d_B = B^{-1} Nd_N$$

$$= \phi + \alpha (c_B^T B^{-1} Nd_N + c_N^T d_N)$$

$$= \phi + \alpha (-\underbrace{c_B^T B^{-1} a_{n_j}} + c_{n_j})$$

$$= \phi + \alpha (-\underbrace{y^T a_{n_j}} + c_{n_j})$$

$n_j^{\text{th}}$  component of

$z_{n_j}$

$$B^T y = c_B$$

$$y = B^{-T} c_B$$

$$y = (c_B^T B^{-1})^T$$

$$z = c - A^T y$$



Sufficient condition.

# Relationship between primal and dual LP.

	finite optimal	unbounded	infeasible
finite optimal			
unbounded			
infeasible			

# Interpretation of dual variables

