Duality

- Primal & dual program
- Strong duality
- Farkas Lemma

Primal and dual programs

- If we transform the dual into an equivalent minimization problem and then form its dual, we obtain a problem equivalent to the original problem

the dual of the dual is the primal.

primal

$$\min_x \; c^T x \qquad \text{dual variable for each constraint} \qquad \max_y \; y^T b$$

s.t. $a_i^T x \geq b_i$	$i \in M_1$	s.t. $y_i \geq 0$	
$a_i^T x \leq b_i$	$i \in M_2$	$y_i \leq 0$	
$a_i^T x = b_i$	$i \in M_3$	y_i - free	
$x_j \geq 0$	$j \in N_1$	$a_j^T y \leq c_j$	
$x_j \leq 0$	$j \in N_2$	$a_j^T y \geq c_j$	
x_j - free	$j \in N_3$	$a_j^T y = c_j$	

Weak/strong duality

<u>Thm:</u> If x is primal feasible and y is dual feasible, then $c^T x \geq b^T y$.

- If the primal optimal cost is $-\infty$, then the dual is infeasible

  Suppose the optimal cost is $-\infty$ and dual has a feasible solution $y \Rightarrow b^T y \leq -\infty$, which is impossible.

- If the dual optimal cost is ∞, then the primal is infeasible.

<u>Thm</u> If a linear programming problem has a optimal solution, so does its dual, and $p^* = d^*$.

Sufficient condition.

Suppose that (x, y, z) is primal/dual feasible.

By weak duality: $x^T c + x^T z = b^T y$

By strong duality, if (x, y, z) is primal/dual optimal:

$$z^T x = 0.$$

conversely: If $z^T x = 0$, then.

- $c^T x$ achieves upper bound
- $b^T y$ achieves lower bound

therefore (x, y, z) is primal/dual optimal.

Theorem: The primal/dual (x, y, z) is optimal iff

$$Ax = b, \quad x \geq 0, \quad A^T y + z = c, \quad z \geq 0, \quad z^T x = 0.$$

Relationship between primal and dual LPs.

Recall: For linear program, exactly one of the following three possibilities hold:

1. There is an optimal solution
2. The problem is unbounded
3. The problem is infeasible.

$c^T x \geq b^T y$

Suppose $p^* = -\infty$ and dual problem has feasible y. By weak duality, $b^T y \leq -\infty$ which is impossible.

dual \ primal	finite optimal	unbounded	infeasible
finite optimal	✓	×	×
unbounded	×	×	✓
infeasible	×	✓	✓

Interpretation of dual variables

primal

$$\min_x \; c^T x$$
$$\text{s.t.} \quad Ax = b$$
$$x \geq 0$$

dual

$$\max_{y,z} \; b^T y$$
$$\text{s.t.} \quad A^T y + z = c$$
$$z \geq 0$$

Suppose x^* is optimal and non-degenerate, then

$$x^* = \begin{bmatrix} B^{-1}b \\ 0 \end{bmatrix} > 0 \quad \text{and} \quad x_B^*(\varepsilon) = B^{-1}(b + \varepsilon \Delta b) > 0$$

for small ε.

$$B^T y = c_B$$

Reduced cost $z^* = c - A^T y^*$ doesn't change. Thus $(x^*(\varepsilon), y^*, z^*)$ is primal/dual optimal and optimal cost is

$$c_B^T x_B = c_B^T B^{-1}(b + \varepsilon \Delta b) = y^T(b + \varepsilon \Delta b) = y^T b + \varepsilon y^T \Delta b$$

Small change of b results in small change in optimal cost.

Certificate of infeasibility

- Consider standard form constraints: $Ax = b$, $x \geq 0$
- Certificate of infeasibility: If there exists some vector y s.t. $A^T y \geq 0$ and $b^T y < 0$, standard form constraint is infeasible.

$$A^T y \geq 0 \Rightarrow x^T A^T y \geq 0 \text{ for all } x \geq 0$$

$$b^T y < 0 \Rightarrow x^T A^T y \neq b^T y \text{ for all } x \geq 0$$

$$\Rightarrow Ax \neq b \quad \text{for all } x \geq 0$$

<u>Farkas Lemma</u>: Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$. Then exactly one of the following two alternatives hold:

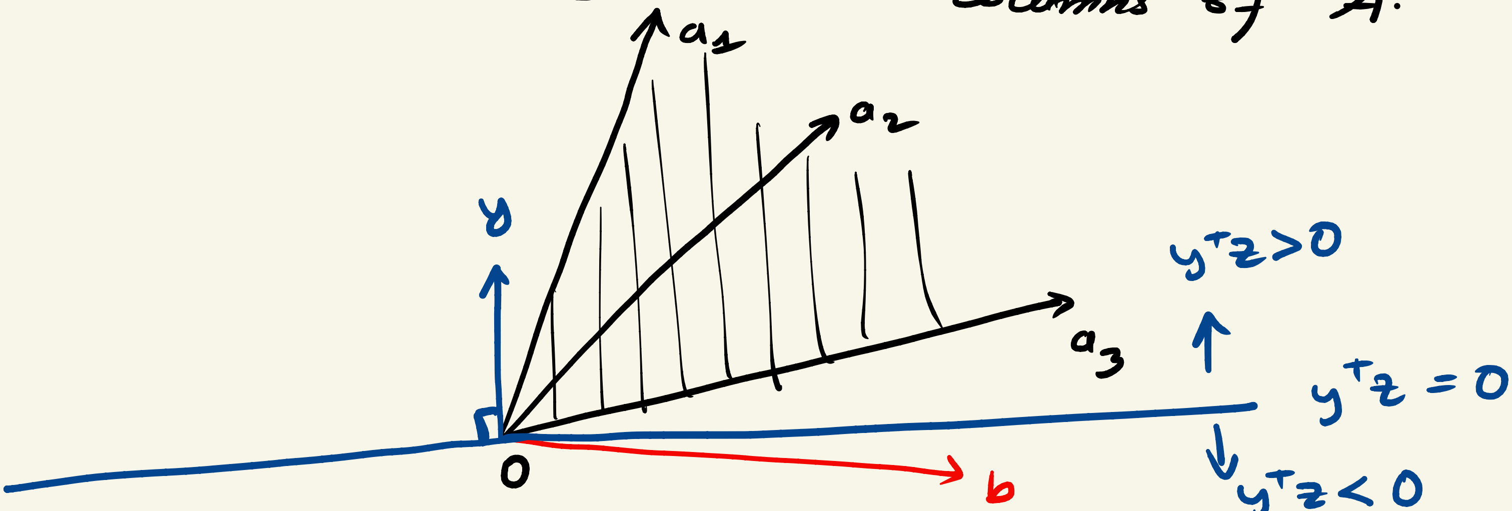
(a) There exists some $x \geq 0$ such that $Ax = b$

(b) There exists some vector y s.t. $A^T y \geq 0$ & $b^T y < 0$.

Geometric view of Farkas' lemma

Let a_i be i^{th} column of A : $A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}$

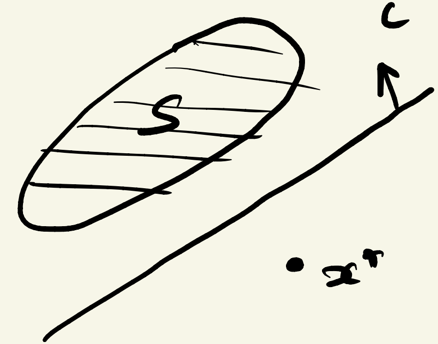
Then $Ax = \sum_{i=1}^{n} a_i x_i$ $\rightarrow$ linear combination of columns of A.



In figure, $Ax \neq b$ for any $x \geq 0$ because the hyperplane $\{z \mid y^Tz = 0\}$ separates b and $\{Ax \mid x \geq 0\}$

Separating hyperplane thm

- Every polyhedron $D = \{x \mid Ax \geq b\}$ is closed.
- <u>Separating hyperplane</u>: Let S be a non empty closed convex subset of $\mathbb{R}^n$ and $x^* \in \mathbb{R}^n$ be a vector that does not belong to S. Then there exists some vector $c \in \mathbb{R}^n$ such that $c^T x^* < c^T x \quad \forall x \in S$.



- $S = \{Ax \mid x \geq 0\}$ is closed

$S' = \{(x,y) \mid Ax = y, x \geq 0\}$ — is a polyhedron

and $S = \text{proj}_y(S')$