Interior point method

History of interior method

The simplex method:

- invented by George Dantzig in 1947
- "walks" the edge of the polyhedral feasible set
- worst-case complexity is exponential (may need to visit every vertex)
- experience (and some analysis) suggest average polynomial complexity

Interior point methods (IPM) are a radical departure from the simplex method:

- IPMs traverses the interior of the polyhedral set
- (impractical) polynomial algorithm for LP first proposed by Kachin (1979)
- Karmarkar (1984) offered first "practical" polynomial LP algorithm.

Big idea

- Constraints are hard to deal with
- Let's turn them into penalties
- We know how to deal with smooth unconstrained problem.

Penalty function

$$\underset{x}{\text{minimize}} \quad f(x) \quad \text{subject to} \quad h_i(x) \geq 0 \quad i=1,\dots,m$$
$$Ax = b$$

- $f, h_i : \mathbb{R}^n \to \mathbb{R}$ are convex and twice differentiable
- $A \in \mathbb{R}^{m \times n}$ with rank $(A) < n$.

Reformulate as

$$\min_x \quad f(x) + \sum_{i=1}^{m} \phi_t(h_i(x)) \quad \text{s.t.} \quad Ax = b$$

<u>Conditions</u>

- $\phi_t(s) \longrightarrow \infty$ as $s \to 0$ → avoid boundary
- $\phi_t(s) \to \infty$ for all $s \geq 0$ as $t \to \infty$ → penalty parameter ??

Indicator Function

First pass example:

$$I_S(x) = \begin{cases} 0 & \text{if } x \in S \\ \infty & \text{if } x \notin S \end{cases}$$

Then, we can reformulate

$$\min_x f(x) \quad \text{s.t. } x \in S \qquad \text{as} \qquad \min_x f(x) + I_S(x)$$

However, this problem is not easy to solve. The objective is not (in general) differentiable.

Eliminating nonnegative constraints

Apply to the primal problem in standard form:

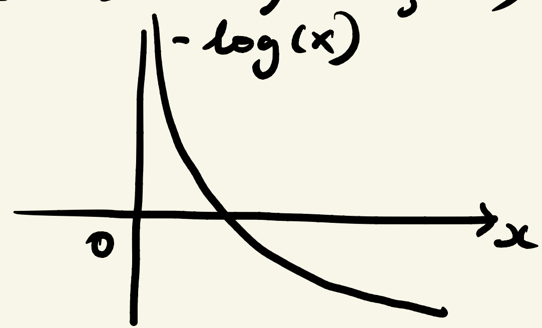
$$\min_x \; c^T x \quad \text{subject to } Ax = b, \quad x \geq 0$$

The core difficulty in LP is the presence of $x \geq 0$.

Eliminate non-negative constraint via barrier function:

$$B_t(x) = c^T x - t \sum_j \log(x_j)$$

- $-\log(x_j) \to \infty$ as $x_j \to 0^+$ (defined as ∞ for $x_j \leq 0$)
- $-t \sum_j \log(x_j) \to \infty$ as $x_j \to 0^+$



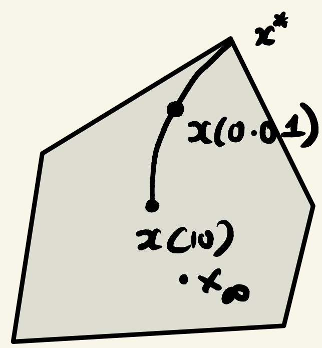
Barrier function

$$(P_t) \qquad \underset{x}{\text{minimize}} \quad B_t(x) \qquad \text{s.t.} \quad Ax = b.$$

- minimizer of the barrier problem depends on t:

  x_t solves P_t.

- minimizer of P_t is unique for each t because of convexity of B_t.



$$x_\infty = \underset{Ax=b}{\arg\min} \; -\sum_i \log(x_i)$$

Example

$$\underset{x}{\text{minimize}} \quad x \quad \text{subject to} \quad x \geq 0$$

$$B_t(x) = x - t\log(x)$$

$$\frac{dB_t(x)}{dx} = 1 - \frac{t}{x} = 0 \Rightarrow x_t = t$$

$$\Rightarrow \lim_{t \to 0^+} x_t = 0$$

Example

$$\underset{x_1, x_2, x_3}{\text{minimize}} \quad x_2 \quad \text{subject to} \quad x_1 + x_2 + x_3 = 1, \quad x \geq 0$$

$$B_t(x) = x_2 - t\log(x_1) - t\log(x_2) - t\log(x_3)$$

$$\Updownarrow$$

$$\underset{x_1, x_2}{\min} \quad x_2 - t\log(x_1) - t\log(x_2) - t\log(1 - x_1 - x_2) =: B_t^2$$

$$\Rightarrow \quad x_1(t) = \frac{1 - x_2(t)}{2} \rightarrow \frac{1}{2}$$

$$x_2(t) = \frac{1 - 2t - \sqrt{1 + 9t^2 + 2t}}{2} \rightarrow 0$$

$$x_3(t) = \frac{1 - x_2(t)}{2} \rightarrow \frac{1}{2}$$

$$X^* = \left\{ (x_1, 0, x_3) \;\middle|\; \begin{array}{c} x_1 + x_3 = 1 \\ x \geq 0 \end{array} \right\}$$

This problem has infinitely many solutions:

Example

$$\min_x \; c^T x \quad \text{subject to } Ax \le b$$

Reformulate as

$$\min_x \; c^T x \underbrace{- t \sum_{i=1}^{m} \log(b_i - a_i^T x)}_{:= \phi_t(x)}$$

Gradient $\nabla \phi_t(x) = t \cdot A^T z$, $z = \dfrac{1}{b_i - a_i^T x}$

Hessian $\nabla^2 \phi_t(x) = t \cdot A^T \operatorname{diag}(z)^2 A$.

- pick some t to start
- solve approximately $\min_x \; c^T x + \phi_t(x)$
- decrease t

Primal barrier method

solve a sequence of linearly constrained nonlinear functions:

choose $x_0 > 0$, $t_0 > 0$ $(\simeq 1)$, $\tau < 1$

repeat

x_{k+1} argmin $B_t(x)$ subject to $Ax = b$

$t_{k+1} \leftarrow \tau t_k$

until t_k is "small"

under mild conditions $x_k \to x^*$.