

## Continuous Compound Interest

MATH 104

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Recall from your high school studies that the compound interest formula is

$$A = P \left( 1 + \frac{r}{n} \right)^{nt},$$

where  $P$  is the principal,  $r$  is the annual interest rate as a fraction,  $n$  is the number of compounding periods per year,  $t$  is the number of years, and  $A$  is the future value at the end of  $t$  years.

For example, if you invest \$1000 at 10% annual interest rate compounded semi-annually (twice per year) for three years, you will have

$$\begin{aligned} A &= 1000 \left( 1 + \frac{0.10}{2} \right)^{2 \cdot 3} \\ &= 1000(1.05)^6 \\ &\approx 1340.10. \end{aligned}$$

That is, your investment will be worth \$1340.10 at the end of three years.

As we increase the number of compounding periods per year, the fact that

$$e = \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n} \right)^n,$$

means that there is a limiting value to our investment, namely,

$$A = Pe^{rt}.$$

In this case, we say that we have *continuously compounded* the interest. While this may seem extreme, the difference between daily compound interest and continuously compounded interest is actually small. For example, if we increase the compounding periods in our example to 365, we would have \$1349.80, and if we compounded continuously, we would have \$1349.86. (Note that this number is not so much greater than when we compounded semi-annually!)

One of the main uses of continuous compound interest is as an easy approximation for the usual discrete compound interest formula. This exponential function, for example, is easily differentiated, and it is easy to compute. This makes it a useful theoretical tool.

We can use this tool to understand the way the value of money changes in time, at least in a simplistic fashion. For example, if there is a 3% annual inflation rate, what is the future purchasing power of \$1000 in 2021 if we assume no other effects? (You might think that you stuck the \$1000 in a drawer and forgot about it until 2021: What could you buy in 2021?) In this case, we will want to think of inflation as *discounting* the value of our money. Thus, the future discounted value,  $A_f$  of our \$1000 is

$$A_f = 1000e^{-0.03 \cdot 10} \approx 740.82.$$

This means that \$1000 in 2021 would have the same purchasing power as \$740.82 dollars in 2011. (Again, we are assuming no other effects except simple inflation.)

We can look at this question a little differently. Suppose you are to receive a graduation gift of \$10000 in four years time. What is the *present value* of this money assuming an inflation rate of 4%? In this instance, we can think of this as asking for the value of the principal  $P$  that should be invested at 4% to end up with \$10000 in four years.

Given  $A = Pe^{rt}$ , we rearrange this to be  $P = Ae^{-rt}$ . Thus, for our case, we have

$$P = 10000e^{-0.04 \cdot 4} \approx 8521.44.$$

This means that the present value of our future \$10000 gift is \$8521.44.

Let's consider another example. Suppose that you purchase a painting today for \$7000 and that you sell it in 50 years for \$7 million. What is the annual rate of return on your art investment (assuming it to be constant over time)?

We wish to solve the equation

$$7000000 = 7000e^{50r}$$

for  $r$ . This gives us

$$r = \frac{1}{50} \ln(1000) \approx 0.138 = 13.8\%.$$

This means our art purchase was a pretty good investment!

Note: For exams, you will have no calculator. You will be asked to leave all answers in *calculator-ready* form. In the last example, this would be  $r = \ln(1000)/50 = 3 \ln(10)/50$ .