MATH 104: Week 5 Learning Goals
Introduction to Finding Maxima and Minima. Optimization Problems. The Second Derivative Test. Concavity. Curve Sketching.

Learning Goals

This week we will introduce Maxima and Minima, which is covered in section 3.5 of the Course Notes. We will also cover the second derivative test, concavity, and curve sketching. We will make use of the first derivative test in this work. This material is in Section 3.6 of the CLP Course Notes. This material will include another look at limits at infinity, infinite limits, and a first look at asymptotes.

Lectures, Readings, Assignments, and Workshops

- **Readings:** In the CLP Notes: Chapter 3.5.1 Local and Global Maxima and Minima, 3.5.2 Finding Global Maxima and Minima. Chapter 3.5.3, Chapter 3.5.4., and Chapter 3.6.
- **Problems:** We encourage you to do some of the problems in each section as you work through it to test your understanding of the material. Answers and solutions to the problems are provided in the text. If the material is new to you, start with the basic problems and work towards more difficult problems. Even doing a small number of problems while you work through the material in the text will help build your understanding.

Detailed Learning Goals:

The specific learning goals for this week are that by the end of the week and review homework, you should be able to:

1. define *absolute maximum* and *absolute minimum* and give examples of functions that illustrate these concepts;
2. state the Extreme Value Theorem, and give examples that illustrate their understanding of this theorem: (1) examples where the EVT applies, and (2) examples where the EVT does not apply, but functions have absolute maxima or minima;
3. define *local maximum* and *local minimum* and give examples of functions that illustrate these concepts;
4. define *critical point* and apply this definition to find and classify critical points of a given function;
5. find the absolute maximum and absolute minimum of a given continuous function on a closed interval.
6. interpret the idea of optimization as the procedure used to make a system or a design as effective or functional as possible, and translate it into a mathematical procedure for finding the maximum/minimum of a function;
7. set up an optimization problem by identifying the *objective function* and all appropriate *constraints*; and
8. use calculus to solve optimization problems, and explain how they used the constraints in the solution process.

9. use calculus to sketch a graph of a given function. Specifically, this means you will be able to
10. explain how the first derivative of a function determines where the function is increasing and decreasing and apply this to specific functions to determine their intervals of increase and decrease;
11. use the first derivative test to identify local maxima and minima;
12. explain how the second derivative of a function determines concavity and apply this to specific functions to determine where they are concave up and concave down, and to identify inflection points;
13. use the second derivative test to classify local maxima and minima;
14. identify any asymptotic behaviours a function may have: vertical asymptotes, horizontal asymptotes, and oblique or slant asymptotes;

Some Food for Thought As You Study This Week

1. We will begin our study of optimization, which will involve the maximum and minimum values of functions. Please read Definition 3.5.6 and surrounding material carefully. There are various other ways that calculus textbooks define **critical points**, for example, and we will want to use language carefully here by sticking to the conventions chosen by the authors of the Course Notes. If you have studied critical points before, you will want to go through the exercise of matching your prior knowledge to the way these ideas are presented in this course. We will stick to the conventions in the Course Notes to keep the confusion to a minimum on language.
2. In dealing with absolute and local maxima and minima, please consider carefully the language used in the Course Notes. There are a lot of terms, and you should undertake the exercise of getting them straight.
3. DRAWING LOTS OF PICTURES helps in this section. The Course Notes are great for this. Be sure to work through examples involving actual functions as well, as pictures are fine, but you need to connect these ideas to working with explicitly defined functions.
4. One difficulty many of you may have at first is setting up optimization problems so you correctly identify the objective function (the function you wish to optimize) and *all* constraints. You should take some time to familiarize yourself with the overall mathematical structure of these problems.
5. It is likely best to start with basic examples when studying the material and working through problems. I suggest working through the exercises from easier to harder problems.
6. Once you have set up the problem, be sure to emphasize how you are using the constraints. This includes *all* the constraints. There is a tendency to downplay or forget constraints like, for example, $x \geq 0$ for some quantity x in a problem because we take it for granted. However, you need to be explicit in how you present this to us in your work, so you should be explicit in how you state and apply such a constraint. Many of the problems involve using one of the constraints to eliminate a variable in the problem to reduce it to a single-variable calculus situation.
7. It is important to understand that simply finding critical points is not the end of solving an optimization problem. You need to show you have found an absolute maximum or minimum in these problems, and will make use of the extreme-value theorem frequently, and one of the first or second derivative tests. You will also need to check things like singular points and end points.

8. The material in 3.6 is presented using the viewpoint "what derivatives tell us." It is crucial to spend time this week making links between the calculations you do and the intuitive aspects of drawing sketches of functions to help you interpret you results. I do recommend that you use a graphing tool (e.g. www.wolframalpha.com) so that students can see how they can check your answers by viewing the final graphs. It will be worthwhile to show them that wolframalpha also has some behaviours that give then unexpected results for functions like $f(x) = x^{2/3}$.

9. It is useful to master "good bookkeeping" techniques for the first derivative test. The kinds of figures that encode the first derivative test (e.g. see the table in Example 3.6.2) are very useful to students to track their results. They are also a good way to present the information clearly and precisely. There are other forms of such tables, but the result is the same. Students who do not track their thinking and express it through these sorts of tables or diagrams tend to get lost and lose many marks on exams.

10. FOR THOSE LOOKING FOR MASTERY OF DEEPER CONCEPTS: Inflection points are worth thinking about carefully. The Course Note's definition of an inflection point is not consistent with the actual geometric definition of such points. For example, there is an inflection point in the curve $y = x^{1/3}$ at $x = 0$ even though neither $f'(0)$ nor $f''(0)$ are defined. There is a vertical tangent line at this point and the curve does change concavity as you pass through this point. Looking at this as the graph of $x = y^3$ may help the students see this.

11. We expect you to be able to find vertical, horizontal, and oblique asymptotes.

12. The curve sketching checklist in section 3.6.5 is nicely presented. We encourage you to practice sketching many curves!