Price elasticity of demand

Common economic definition:

The percentage change in quantity demanded divided by the percentage change in price, i.e.

$$\varepsilon = \frac{\%\,\Delta q}{\%\,\Delta p} = \frac{p}{q} \cdot \frac{\Delta q}{\Delta p} \quad \left(\%\,\Delta q = \frac{\Delta q}{q} \times 100 \right)$$

Formal definition: $\varepsilon = \dfrac{p}{q} \dfrac{dq}{dp}$

What is the sign of ε?

The law of demand says that $p \uparrow$, $q \downarrow$ i.e.

$$\frac{dq}{dp} < 0$$

So, the price elasticity of demand is negative, $\varepsilon < 0$.

Why do we care about price elasticity of demand?

Q. How will revenue change as we adjust price?

$$\text{Revenue} = p \cdot q, \quad R = p \cdot q(p)$$

If price increases, quantity demanded decreases.

But it's unclear if revenue will increase/decrease.

Change in Revenue

The increase/decrease of revenue depends on the relative change in p and q.

eg: If $p_{new} = 1.2p$ (20% increase)

$q_{new} = 0.99q$ (1% decrease)

Then, $R_{new} = p_{new} \cdot q_{new} = 1.2p \times 0.99q = \underbrace{(1.2 \times 0.99)}_{>1} \underbrace{pq}_{R}$

So, revenue increase

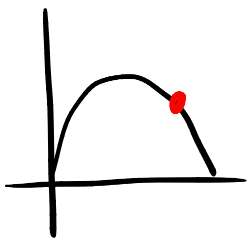
If $p_{new} = 1.2p$, $q_{new} = 0.5q$ then

$R_{new} = \underbrace{(1.2 \times 0.5)}_{<1} \underbrace{p \cdot q}_{R}$

So, revenue decreases.

Revenue and price elasticity

What is the relationship between price elasticity of demand and revenue?



$$R = p \cdot q(p) \quad (\text{treat } q = q(p))$$

$$\frac{dR}{dp} = 1 \cdot q(p) + p \cdot \frac{dq}{dp} \quad (\text{product rule})$$

$$= q\left(1 + \frac{p}{q}\frac{dq}{dp}\right)$$

$$= q(1+\varepsilon) \quad (\text{definition of } \varepsilon).$$

Note that q is always positive. So, the sign of $\frac{dR}{dp}$ is determined by $1+\varepsilon$, i.e., how large in absolute value ε is relative to 1.

Three cases of $(1+\varepsilon)$

Case 1, $|\varepsilon|>1$: We say the good is price elastic. | $\varepsilon = \frac{\%\Delta q}{\%\Delta P}$

Here, $|\%\Delta q| > |\%\Delta p|$. So, a 1% increase in price leads to a greater than 1% decrease in quantity demand.

Mangement should decrease price to increase revenue

$$\frac{dR}{dp} = q(1+\varepsilon)$$

Case 2, $|\varepsilon|<1$: We say the good is price inelastic.

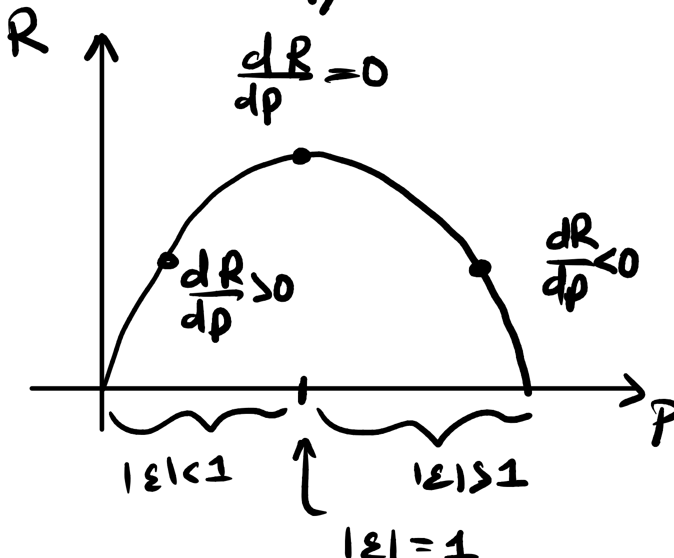
Here, $|\%\Delta q| < |\%\Delta p|$. So, a 1% increase in price leads to a less than 1% decrease in quantity demanded.

Increase price to increase revenue.

Case 3, $|\varepsilon| = 1$ we say the good is **price unit elastic.**

Here, $|\%\Delta q| = |\%\Delta p|$. So, a 1% change in price causes a 1% decrease in quantity.

This is the optimal price to maximize revenue (R is maximized when $\frac{dR}{dp} = q(1+\varepsilon) = 0$)



For a linear demand:

$q(p) = -ap + b, \quad a, b > 0$

$R = p \cdot q$

$\quad = p(-ap + b)$

quadratic function.

Example

Suppose the demand curve of OPads is given by $q = 500 - 10p$.

(a) Compute ε (price elasticity of demand).

$$\varepsilon = \frac{p}{q} \cdot \frac{dq}{dp} = \frac{p}{q} \times (-10) = \frac{-10p}{500 - 10p} = \frac{p}{p - 50}$$

$$\varepsilon = \frac{\% \Delta q}{\% \Delta p}$$

(b) What is the price elasticity of demand when $p = \$30$.

From (a) = $\varepsilon(30) = \dfrac{30}{30 - 50} = -\dfrac{3}{2} = -1.5$

Since $|\varepsilon| > 1$, the good is price elastic and OPad should decrease price to increase revenue.

Example contd.

c) What is the percentage change in demand if the $p = \$30$ and increase by 4.5%.

$$\varepsilon = \frac{p}{q}\frac{dq}{dp} \approx \frac{\%\Delta q}{\%\Delta p}$$

Hence $\frac{\%\Delta q}{\%\Delta p} = -1.5 \Rightarrow \%\Delta q = -1.5 \times 4.5\% = -6.75\%$

So, demand is decreased by 6.75%.

ⓓ How does the change in demand inform about the change in price?

decrease price to increase revenue because $|6.75\%| > |4.5\%|$

Another example to compute ε

$f'(x)$ is the rate of change of f wrt x.

$\dfrac{f'(x)}{f(x)}$ is the relative rate of change of f

Note $\dfrac{f'(x)}{f(x)} = \dfrac{d}{dx} \ln(f(x))$ by chain rule

$$\varepsilon = \frac{\text{relative rate of change of } q \text{ wrt. } p}{\text{relative rate of change of } p \text{ wrt. } p}$$

$$= \frac{\frac{d}{dp} \ln(q(p))}{\frac{d}{dp} \ln(p)} = \frac{\frac{1}{q(p)} \cdot q'(p)}{\frac{1}{p}} = \frac{p}{q} \cdot \frac{dq}{dp} \equiv \varepsilon.$$

Continuous compound interest.

Ex Suppose you have \$100 in a bank with an annual interest rate of 100%. How much money will you have in the bank after 1 year if the interest is compounded.

(a) Once a year:

Soln $\$100(1+1) = \$200$

(b) **Semi annually:**

Soln $\$100\left(1+\frac{1}{2}\right)\left(1+\frac{1}{2}\right) = \$100\left(1+\frac{1}{2}\right)^2 \approx \$225$

(c) **Every day of year:**

Soln: $\$100\left(1+\frac{1}{365}\right)^{365} \approx \$271.46$

Compound interest

d) Compound n times in a year: $\$100\left(1+\frac{1}{n}\right)^n$

If we take $n \to \infty$: $\lim_{n\to\infty} 100\left(1+\frac{1}{n}\right)^n$

$$= 100 \underbrace{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}_{\text{This turns out to equal } e \approx 2.718\ldots \text{ Euler's number.}} = 10 \cdot e^{1.1}$$

$\approx \$271.8$

In this case we say the interest was compounded continuously.

Compound interest

In general, the compound interest formula is:

$$A = P\underbrace{\left(1+\frac{r}{n}\right)^{rt}}$$

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{rt} = e^{rt}$$

P = Principal (initial investment)

r = annual rate of interest.

n = # of compounding periods per year

t = # of years.

A = future value at the end of t years.

As we take $n \to \infty$, $\boxed{A = Pe^{rt}}$

This is called continuously compounding interest.

Example

Find the present value of \$5000 to be recieved in 2 years if the money can be invested at 12% annual interest rate compounded continuously.

Sol^n Given $A = 5000$, $r = 0.12$, $t = 2$

Find P

$$A = Pe^{rt} \iff P = Ae^{-rt}$$

$$= 5000\, e^{-0.12 \times 2}$$

$$\approx 3933.14$$

Population growth

Population growth can be modeled as:

$$P(t) = P_0 e^{rt}$$

$P(t)$ is population after time t

P_0 is initial population.

r is the rate of population growth.

t is time

Eg: In 1927, the population of the world was ~ 2 billion. In 1974, the population was ~ 4 billion.

Estimate the time when the population reaches 6 billion.

Example sol^n

Let $P(t)$ be population in t years:

Then $P(0) = P_0 = 2$ (billion).

Hence $P(t) = 2e^{rt}$

Find r: when $t = 1974 - 1927 = 47$ years.

$$P(47) = 2e^{r47} = 4$$

$$\Rightarrow e^{47r} = 2 \Rightarrow r = \frac{\ln(2)}{47}$$

Predict when $P(t) = 6$

$$6 = 2e^{\frac{\ln(2)}{47} \cdot t}$$

$$\Rightarrow 3 = e^{\ln(2)/47 \cdot t} \Rightarrow t = 47 \cdot \frac{\ln(3)}{\ln(2)} \approx 74 \text{ years.}$$

So, Model predicts population is 6 billion in ~~2001~~ 1999.

Exponential model

In general, exponential model is of the form:

$$y(t) = C e^{kt} + D, \quad C, k \text{ are constants.}$$

D

$$y'(t) = kCe^{kt} \quad \text{(chain rule).}$$

$$\boxed{y'(t) = k\, y(t).}$$