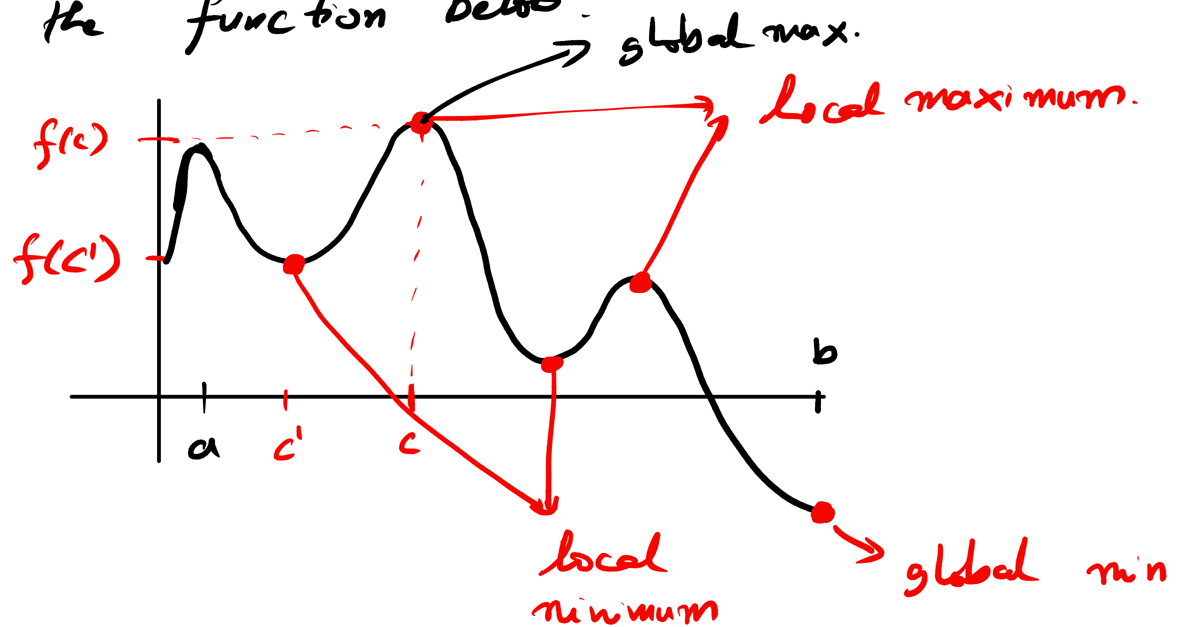
Optimization

Goal: Find the maximum / minimum of $y = f(x)$ in on interval $a \le x \le b$

Consider the function below:



Classification of min / max

In general, for a function $f(x)$ defined on $[a,b]$,

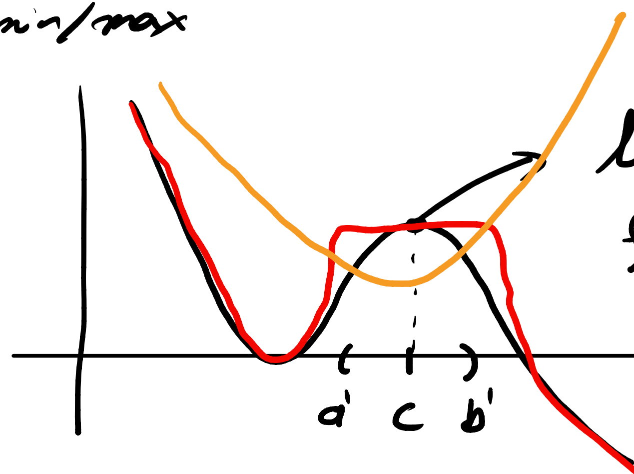
we call a point $c \in [a,b]$:

① A global maximum (absolute maximum) if $f(c) \geq f(x)$ for all $x \in [a,b]$.

② A global (absolute) minimum if $f(c) \leq f(x)$ for all $x \in [a,b]$

③ A local maximum if $a < c < b$ and we can find a (small) interval (a', b') containing c where $f(c) \geq f(x)$ for all x in (a', b')

Classification of min/max



local max: $f(c) \geq f(x)$ for all $x \in (a', b')$

4. A local minimum if $a < c < b$ and we can find a small interval (a', b') around c where $f(c) \leq f(x)$ for all $x \in (a', b')$.

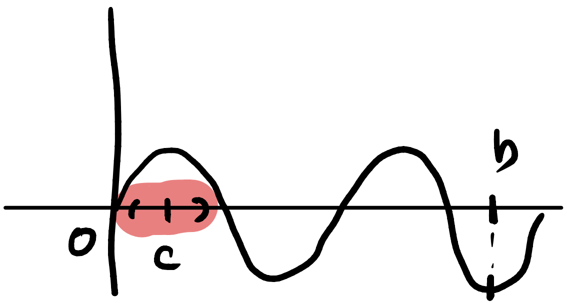
Local extrema = local maximum or minimum

Global extrema = global max or min.

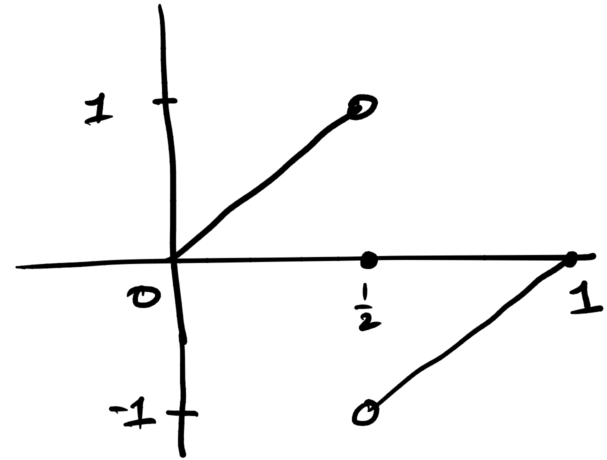
<u>Note</u>: The endpoints of $[a, b]$ are never local max/min extremum point

Extreme value theorem

Q: Does every function defined on $[a,b]$ have a global max/min ?



$f(c) \geq f(x)$, for all $x \in (a', b')$



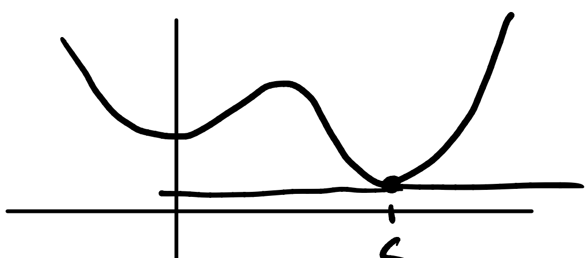
function has no global max or global min.

Extreme Value theorem

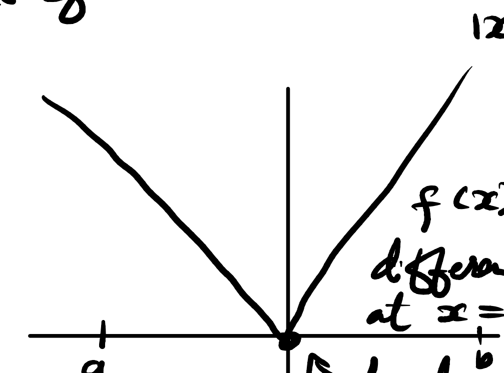
Thm If $f(x)$ is a continuous function on $[a,b]$ then $f(x)$ has atleast one global minimum and at least one global maximum in the interval $[a,b]$.

How do we find the global extrema?

Global extrema can occur at endpoints of $[a,b]$ or at local extrema.



$x=c$ satisfies $f'(c)=0$ (critical point)



$f(x)$ is not differentiable at $x=0$

local min is attained at singular point

Def^n: Let $f(x)$ be a function.

1. $x=c$ is a critical point if $f'(c)=0$
2. $x=c$ is a singular point if $f'(c)$ does not exist.

Thm If $x=c$ is a local extremum then $x=c$ is either a critical point or a singular point

Q: How to find global extrema?

1. Make a list of all values c in $[a,b]$ for which:
   - (a) c is a critical point
   - (b) c is a singular point
   - (c) $c=a$ or $c=b$ (end points)

2. Evaluate $f(c)$ for all values on the list and compare.

Example:

Find all global extrema of $f(x) = x^2 + 6x - 10$ on $[-5, 5]$.

- Critical points: $f'(x) = 2x + 6 = 0 \Rightarrow x = -3$
- Singular points: No singular points
- End points:

So, global min/max can occur at $x = -3, -5, 5$

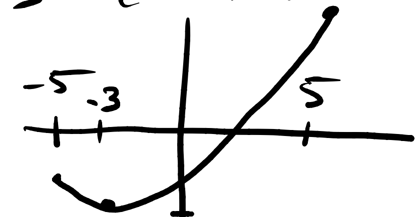
Compare

$$f(-3) = 9 - 18 - 10 = -19 \longrightarrow \text{global min}$$

$$f(-5) = 25 - 30 - 10 = -15$$

$$f(5) = 25 + 30 - 10 = 45 \longrightarrow \text{global max.}$$

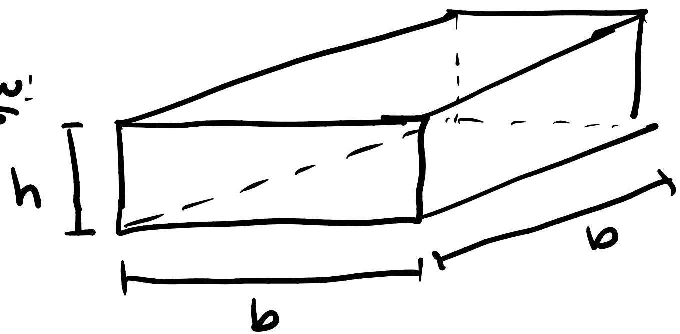
Hence $f(x)$ has a global min at $x = -3$ & max at $x = 5$.



Example

A closed rectangular container with a square base is to be made from two different materials. The materials of the base (bottom) cost $\$5/m^2$, the material of other sides cost $\$1/m^2$. Find the dimensions which maximizes volume if total cost of the material is $\$72$.

Draw:



- constraint

$h \geq 0$, $b \geq 0$

Volume ≥ 0

Cost $= 72$

Want to maximize volume

$h =$ height, $b =$ base (in meters)

Example

$$\text{Volume} = V = b^2 h \quad \text{function of } b, h$$

$$\text{Total cost} = \underline{\underline{72}} = 5b^2 + 1\times(4b\cdot h + b^2) = \underline{\underline{6b^2 + 4bh}}$$

$$\Rightarrow \quad 72 = 6b^2 + 4bh$$

$$\Rightarrow \quad h = \frac{72 - 6b^2}{4b} = \frac{36 - 3b^2}{2b}$$

$$\text{so, } V = b^2 \cdot \frac{36 - 3b^2}{2b} = \frac{(36 - 3b^2)\cdot b}{2} = 18b - \frac{3}{2}b^3$$

To find end points, $V \geq 0 \Rightarrow \dfrac{(36 - 3b^2)\cdot b}{2} \geq 0$

$$\Rightarrow \quad b \in [0, \sqrt{12}]$$

Critical points: $V'(b) = 18 - \dfrac{9}{2}b^2 = 0$

$$\Rightarrow \quad \frac{9}{2}b^2 = 18 \Rightarrow b^2 = 4 \Rightarrow b = 2$$

Compare $V(0) = 0$, $V(\sqrt{12}) = 0$ endpoints.

$V(2) = 36 - \cancel{24} = 24\ m^3$
3×4

Hence, to maximize volume while maintaining cost constraint, we need $b = 2m$ and

$$h = \frac{36 - 3b^2}{2b} = \frac{36 - 3\cdot 4}{2\cdot 2} = 6m$$

$2m \times 2m \times 6m$

Elasticity, CLP 3.3, 2.13, 3.2, 3.5, 2.12