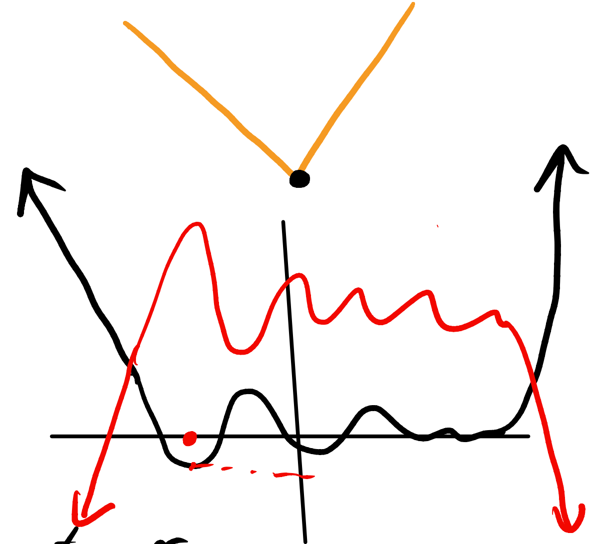
Functions without restrictions

Thm Let $f(x)$ be continuous for all $-\infty < x < \infty$.



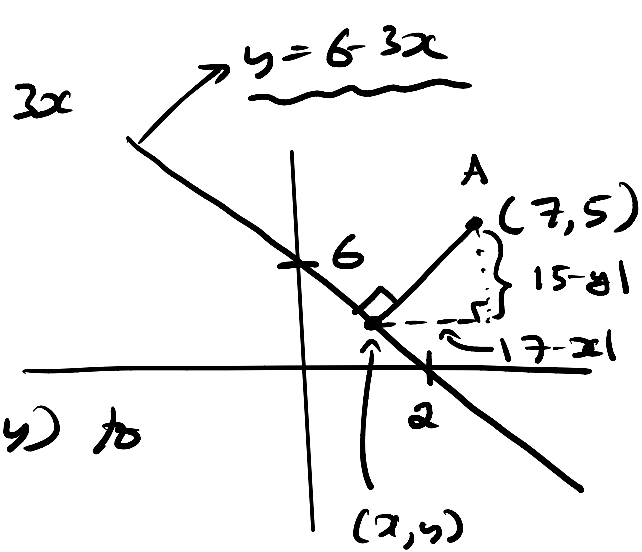
① If $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = \infty$

then $f(x)$ has a global minimum. and it occurs at either a critical point or a singular point.

② If $\lim_{x\to\infty} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = -\infty$, then $f(x)$ has a global maximum and it . . . . .

Example

Q. Find the point on the line $y = 6 - 3x$ that is closest to $(7, 5)$



$(1, 6-3)$

$(1, 3)$

Let (x, y) be a point on the line

Let L be the distance from (x, y) to $(7, 5)$

Goal: minimize L.

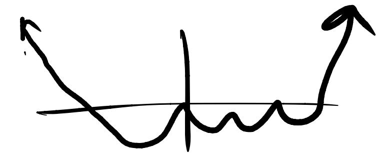
$L \geq 0$

$$L(x, y) = \sqrt{(7-x)^2 + (5-y)^2},$$
$$= \sqrt{(x-7)^2 + (y-5)^2}$$

constrainted: $y = 6 - 3x$. No restriction x!

Example contd.

$y = 6 - 3x$



$$L(x) = \sqrt{(x-7)^2 + (\underbrace{6-3x-5}_{(1-3x)})^2}$$

$$= \sqrt{x^2 - 14x + 49 + (1 - 6x + 9x^2)}$$

$$= \sqrt{10x^2 - 20x + 50} = \boxed{\sqrt{10}\sqrt{x^2 - 2x + 5}} \to \sqrt{10}(x^2-2x+5)^{1/2}$$

Check: $\lim_{x \to \infty} L(x) = \infty$, $\lim_{x \to -\infty} L(x) = \infty$

So, by previous theorem, global minimum exists and it occurs at a singular point or a critical point.

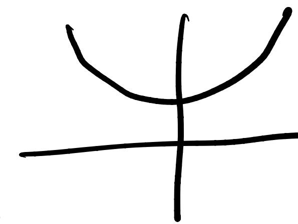
$$L'(x) = \sqrt{10}\,\frac{1}{\cancel{2}(x^2-2x+5)^{1/2}}(\cancel{2}x - \cancel{2}^{\,1}) = \frac{\sqrt{10}\,(x-1)}{(x^2-2x+5)^{1/2}}$$

No singular points exists because $x^2 - 2x + 5 \neq 0$ for all x.

By quadratic formula, the zeros of x^2-2x+5 are

$$x = \frac{2 \pm \sqrt{4-10}}{2} = \frac{2 \pm \sqrt{-6}}{2}$$

Discriminant of x^2-2x+5 is negative $\Rightarrow$ x^2-2x+5 does not have any real roots.



<u>critical points:</u> $L'(x)=0$

$$\Rightarrow \frac{\sqrt{10}(x-1)}{(x^2-2x+5)^{1/2}} = 0 \Rightarrow x = 1.$$

and $y = 6-3\cdot 1 \Rightarrow 3.$

and the smallest distance is

$$L(x) = \sqrt{10}\sqrt{x^2-2x+5}$$

$$L(1) = \sqrt{10}\sqrt{1-2+5}$$

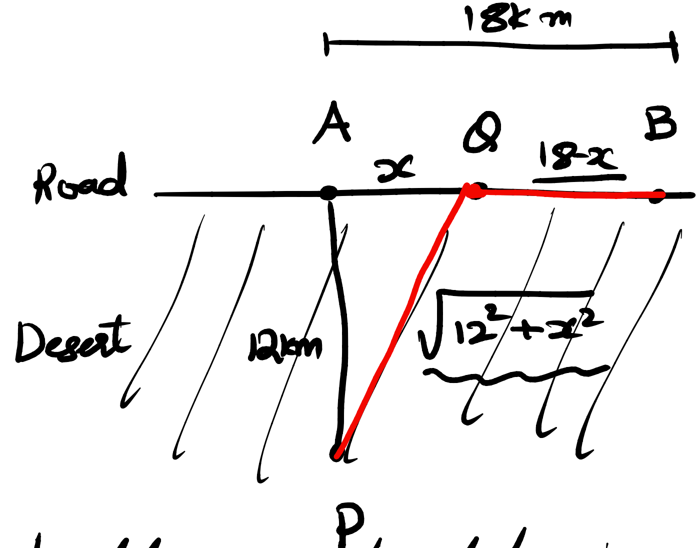
$$= 2\sqrt{10}$$

The closest point on the line is $(\underline{1,3})$

Example

speed on desert is 15 km/hr
speed on road is 30 km/hr.

distance from A to P is 12 km.
distance from A to B is 18 km.



Q: What point Q on the road should you head to in order to minimize travel time?

Let x be the distance from A to Q.

Let $T(x)$ be the total time taken.

$$T(x) = \text{time on desert} + \text{time on road}.$$

$$= \frac{\sqrt{144+x^2}}{15} + \frac{18-x}{30}, \quad x \in [0, 18].$$

Example contd.

We need to minimize $T(x)$ for $x \in [0,18]$.

$$T'(x) = \frac{d}{dx}\left(\frac{\sqrt{144+x^2}}{15} + \frac{18-x}{30}\right) = \underbrace{\frac{1}{15}\cdot\frac{2x}{2\sqrt{144+x^2}}}_{} + \frac{1}{30}(-1)$$

$$= \frac{x}{15\sqrt{144+x^2}} - \frac{1}{30}$$

No singular points (on $[0,18]$)

$$T'(x) = 0 \Rightarrow \frac{x}{15\sqrt{144+x^2}} - \frac{1}{30} = 0 \Rightarrow \frac{x}{\sqrt{144+x^2}} = \frac{1}{2}$$

$$\Rightarrow 2x = \sqrt{144+x^2}$$

$$3x^2 = 12\cdot 12 \Rightarrow x^2 = 4\cdot 4\cdot 3 \Longleftrightarrow \Rightarrow 4x^2 = 144+x^2$$

$$\Rightarrow x = 4\sqrt{3} \qquad \Rightarrow x = \underbrace{4\sqrt{3}}_{}$$

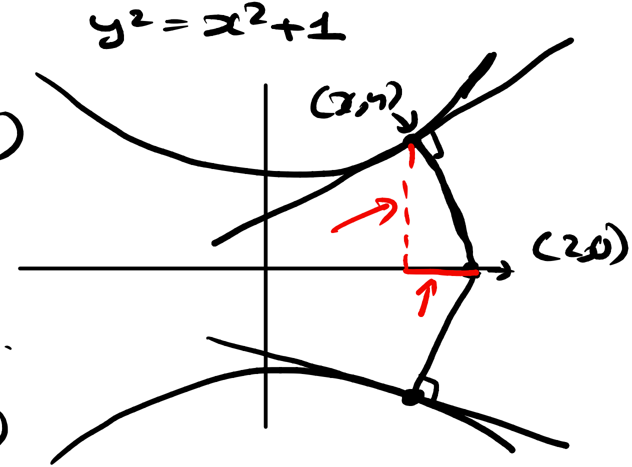
<u>End points:</u> $x = 0$, $x = 18$.

minimized!

$T(0) \approx 1.4$, $T(18) \approx 1.44$, $T(4\sqrt{3}) \approx 1.29$.

Example

Find the minimum distance from $(2,0)$ to the curve $y^2 = x^2+1$.



Let (x,y) be a point on the curve.
Let L be the distance from (x,y) to $(2,0)$.

$$L(x,y) = \sqrt{(x-2)^2 + y^2}$$

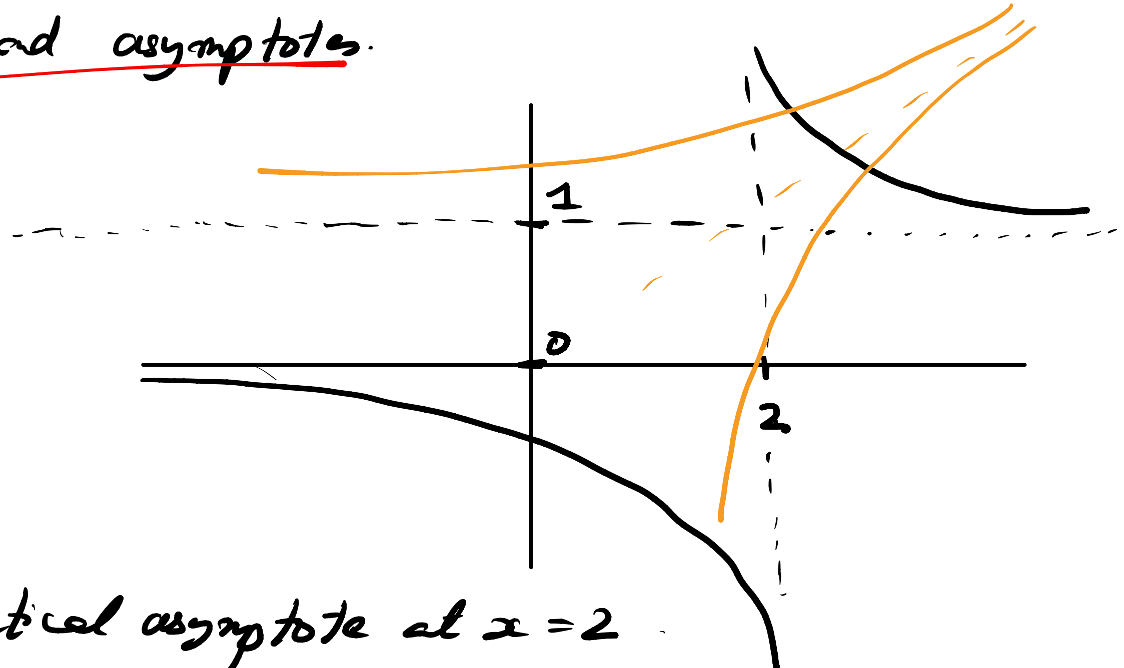
we want to minimize $L(x,y)$ given $y^2 = x^2+1$.

we can minimize $L(x) = \sqrt{(x-2)^2+x^2+1}$ $\quad x \in (-\infty,\infty)$

Let $\tilde{L}(x) = (L(x))^2 = (x-2)^2 + x^2+1$

Curve sketching and asymptotes.

Consider:



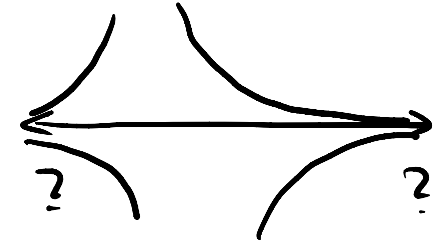
$f(x)$ has a vertical asymptote at $x=2$

horizontal asymptote at $y=0, 1$.

Asymptotes are lines which $y=f(x)$ gets closer and closer to as one or both x and y goes to infinity

Example

Q) Let $g(x) = \dfrac{x-4}{(x-1)(x+2)}$



(a) What is the domain of $g(x)$?

We look for values where $g(x)$ is defined.

$g(x)$ is not defined at $x = 1, -2$.

domain is $(-\infty,-2) \cup (-2,1) \cup (1,\infty)$.

(b) Compute the horizontal and vertical asymptotes of $g(x)$.

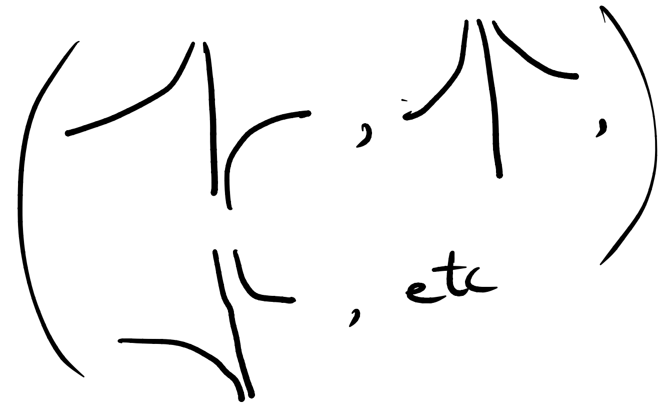
Horizontal: $\lim_{x\to\infty} \dfrac{x-4}{(x-1)(x+2)} = 0$ since denominator has a higher degree polynomial $\Rightarrow$ $y=0$ is a horizontal asymptote.

Similarly $\lim_{x\to-\infty} \dfrac{x-4}{(x-1)(x+2)} = 0 \Rightarrow$ ″

Example contd.

Vertical asymptotes: $g(x)$ has vertical asymptotes at $x=1$ and $x=-2$

ⓒ For each vertical asymptote, determine the shape around it.



At $x=1$

$$\lim_{x \to 1^-} \frac{x-4}{(x-1)(x+2)} = \infty$$

since $\dfrac{x-4}{(x-1)(x+2)}$ look like $\dfrac{\text{neg}}{\text{neg} \cdot \text{pos}}$ if $x \to 1^-$

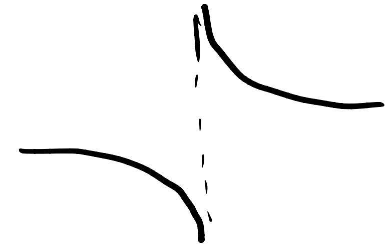
$$\lim_{x \to 1^+} \frac{x-4}{(x-1)(x+2)} \simeq \frac{\text{neg}}{\text{pos} \cdot \text{pos}} \simeq -\infty$$

$x=1$

Example contd.

$$\lim_{x \to -2^-} \frac{x-4}{(x-1)(x+2)} = -\infty$$

$$\lim_{x \to -2^+} \frac{x-4}{(x-1)(x+2)} = +\infty$$



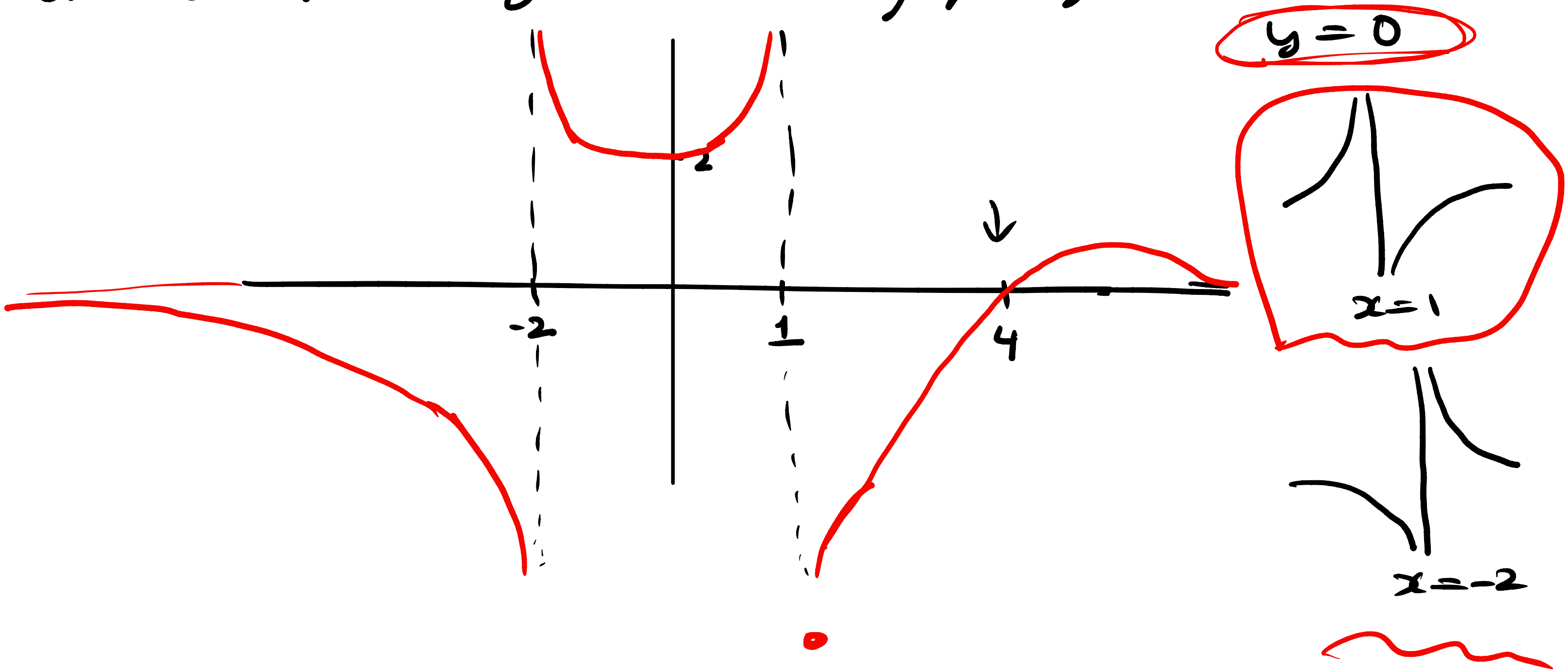
$x = -2$

(d) Compute the x-intercept and y-intercept.

x-intercept, $g(x) = 0 \Rightarrow \dfrac{x-4}{(x-1)(x+2)} = 0 \Rightarrow x = 4$

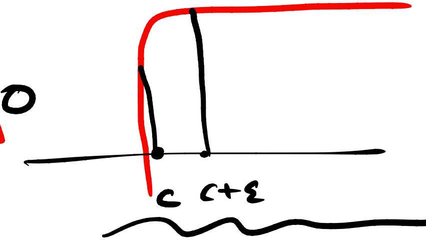
y-intercept, $y = g(0) \Rightarrow y = 2$

(e) Use all the information to graph function.



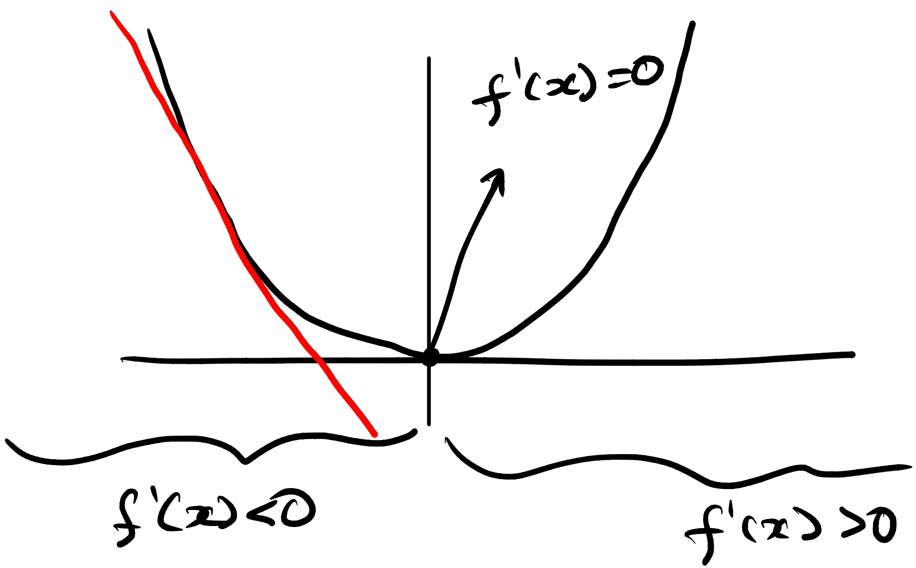
First derivative - Increasing/decreasing

- $f(x)$ increases at $x=c$ if $f'(c) \overset{>}{\boxed{\geq}} 0$



  $f(c) \leq f(c+\varepsilon)$ for $\varepsilon > 0$ and small

- $f(x)$ decreases at $x=c$ if $f'(c) \leq 0$



Example.

Consider the function $f(x) = x^4 - 6x^3$

(a) What is the domain of the function?

domain = all real numbers.

(b) Determine any asymptotes.

No vertical or horizontal asymptotes.

(c) Intercepts:

y- intercept : $y = f(0) = 0$

x- intercepts : $f(x) = 0$

$\Rightarrow x^3(x-6) = 0$

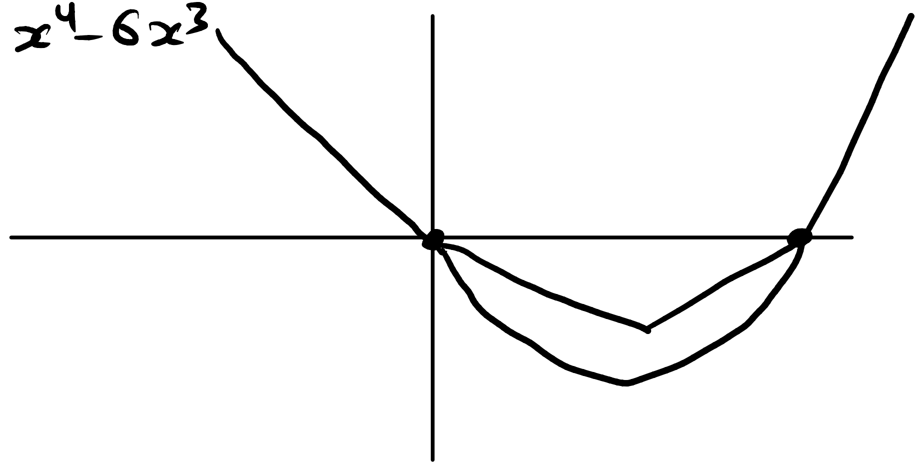
$\Rightarrow x = 0, 6$

(d) Determine where $f(x)$ is positive or negative.

Since $f(x)$ is continuous, only change signs at x-intercepts.

interval	$(-\infty, 0)$	0	$(0, 6)$	6	$(6, \infty)$
$f(x)$	+ve	0	−ve	0	+ve

$f(x) = x^4 - 6x^3$



Example contd.

(e) Determine the singular / critical points.

$f(x) = x^4 - 6x^3$, $\quad \frac{d}{dx} f(x) = 4x^3 - 18x^2$

No singular points.

Critical points: $f'(x) = 0 \Rightarrow 2x^2(2x - 9) = 0$

$\Rightarrow \underline{x = 0}, \; x = 9/2$

(f) Determine the interval where $f(x)$ is increasing/decreasing

interval	$(-\infty, 0)$	0	$(0, 9/2)$	$9/2$	$(9/2, \infty)$
$f'(x)$	$-$ve	0	$-$ve	0	$+$ve
	decreasing		decreasing		increasing.