Second derivatives – concavity

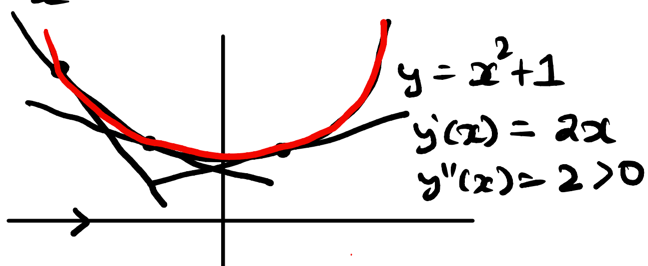
The second derivative $f''(x)$ tells us the rate in which $f'(x)$ changes. $f''(x) = \frac{d}{dx} f'(x)$

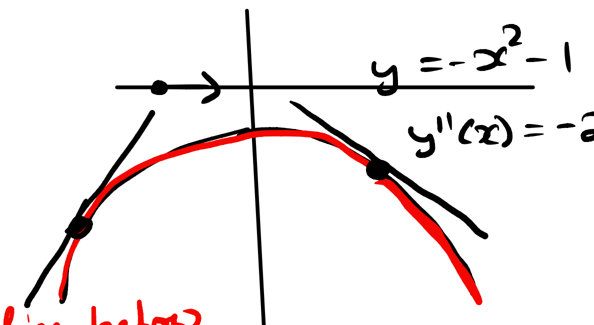
$$\boxed{f'(x) = \frac{d}{dx} f(x)}$$

So, if $f''(x) > 0$, then $f'(x)$ is increasing. Equivalently, the slope of tangent line is increasing as x increases.

if $f''(x) < 0$, then the slope of tangent line is decreasing as x increases.

Ex:





graph lies below the tangent line

concave up

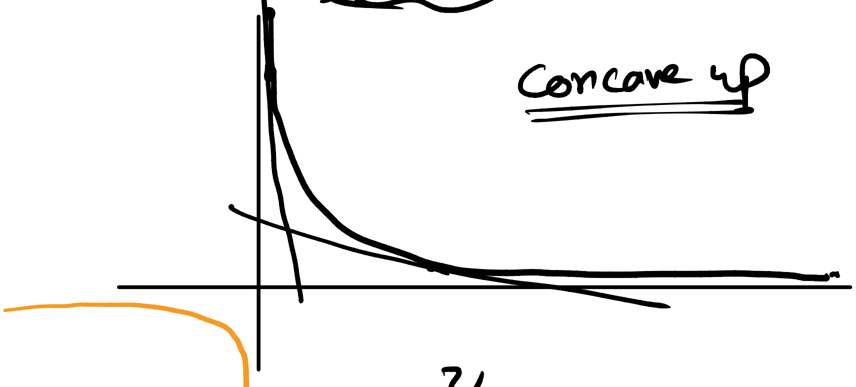
concave down

Example

$(-\infty, 0) \cup (0, \infty)$

consider $y = \dfrac{1}{\sqrt{x}} = x^{-1/2}$

for $x > 0$ $\quad (x<0)$



Concave up

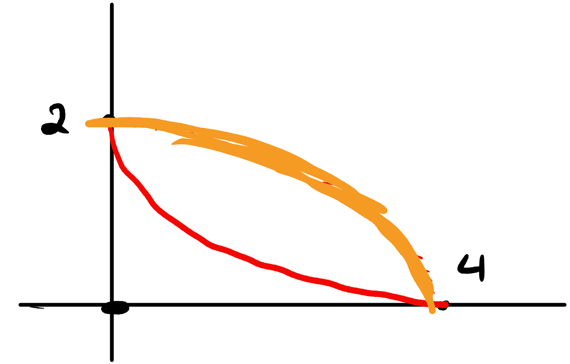
$$y'(x) = -\frac{1}{2} x^{-3/2}$$

$$y''(x) = -\frac{1}{2}\left(-\frac{3}{2}\right) x^{-5/2}$$

$$= \frac{3}{4} x^{-5/2} > 0 \text{ for all } x > 0$$

<0

consider $y = \sqrt{4-x}$.



$$y = (4-x)^{1/2}$$

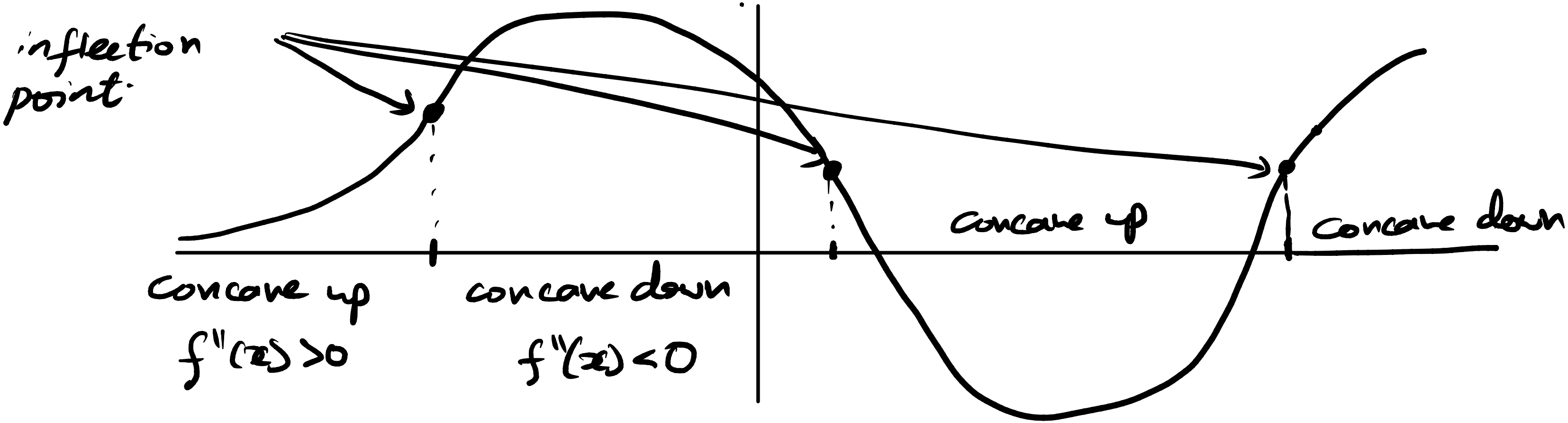
$$y'(x) = \frac{1}{2}(4-x)^{-1/2}(-1)$$

$$y''(x) = -\frac{1}{2}\cdot\left(-\frac{1}{2}\right)(4-x)^{-3/2}(-1)$$

$$= -\frac{1}{4}(4-x)^{-3/2}$$

<0 for $x \in (0,4)$

Concavity and inflection points



$f(x)$ is concave up on $[a,b]$ if $f''(x)>0$ for all $x\in(a,b)$

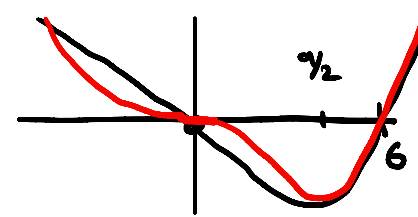
$f(x)$ is concave down on $[a,b]$ if $f''(x)<0$ for all $x\in(a,b)$.

$x=c$ is an inflection point of $f(x)$ if $f''(c)=0$.

The concavity changes at $x=c$.

Example

Let $f(x) = x^4 - 6x^3$



ⓐ Compute the concavity of the function, noting any inflection points.

$\underbrace{f'(x) = 4x^3 - 18x^2}$, $f''(x) = 12x^2 - 36x = \underline{12x(x-3)}$

$f''(x)$ can only change sign when $f''(x) = 0$.

so, $12x(x-3) = 0 \Rightarrow \underline{x = 0}$ or $\underline{x = 3}$.

Divide x-axis at these points:

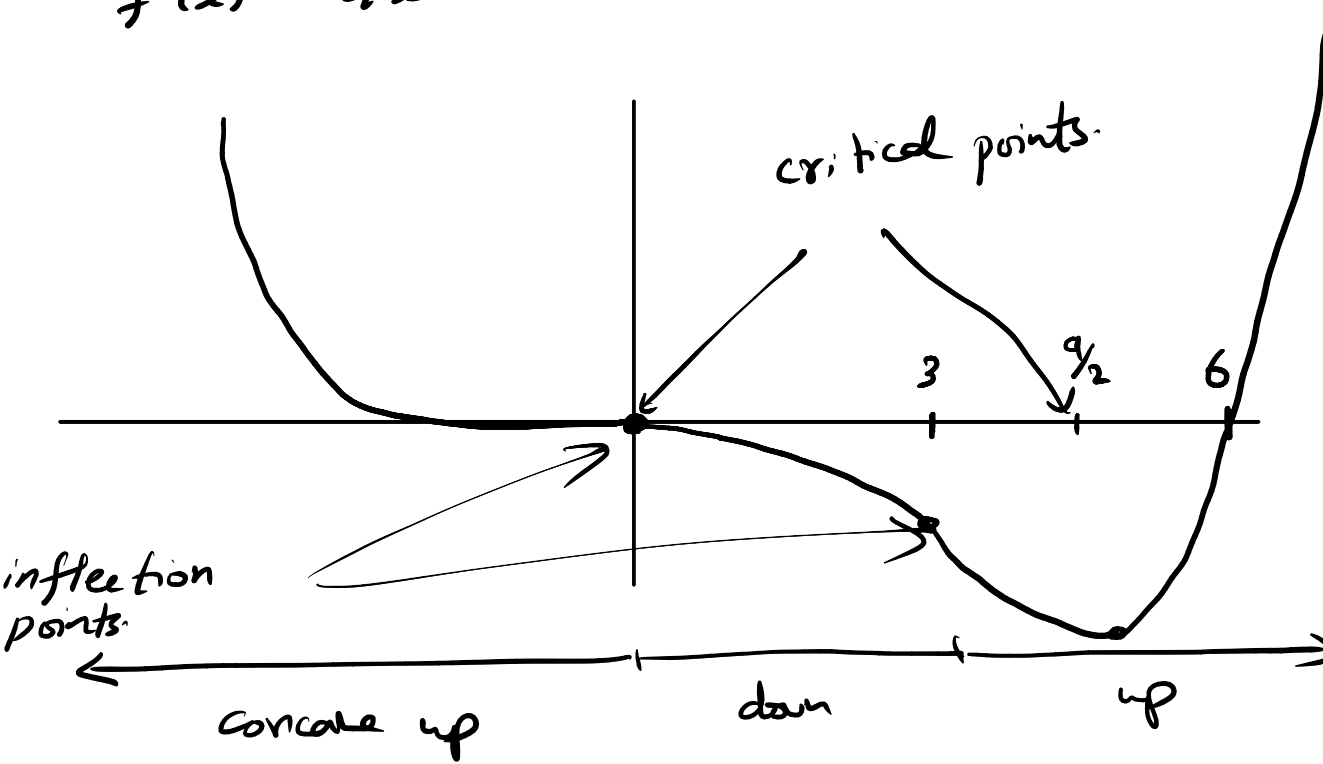
intervals	$(-\infty, 0)$	0	$(0, 3)$	3	$(3, \infty)$
$f''(x)$	+ve	0	-ve	0	+ve
concavity	up		down		up

Example contd.

Note: $f(x) = x^4 - 6x^4 \Rightarrow$ x-intercept $= 0, 6$

y-intercept $= 0$

$f'(x) = 4x^3 - 18x^2 \Rightarrow x = 0, 9/2$ are critical points.



Example

consider $f(x) = x e^{-x^2/2}$

$f'(x) = e^{-x^2/2} + x\left(\frac{-2x}{2}\right) e^{-x^2/2} = (1-x^2) e^{-x^2/2}$

$f''(x) = -x e^{-x^2/2} - 2x e^{-x^2/2} - x^2\left(-2x/2\right) e^{-x^2/2} = (x^3-3x) e^{-x^2/2}$

Note: $\lim_{x \to \pm\infty} f(x) = 0$ (horizontal asymptote)

Find all inflection points to find intervals where function is convex up and convex down

Note: convex up = concave down
convex down = concave up.

Convexity/concavity: Inflection points are:

$f''(x) = 0 \Rightarrow (x^3 - 3x) e^{-x^2/2} = 0 \Rightarrow \underbrace{x = 0, \pm\sqrt{3}}$

$f''(-10) = ((-10)^3 - 3(-10)) e^{-100/2} < 0$, $f(-1) > 0$, ...

interval	$(-\infty,-\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3},0)$	0	$(0,\sqrt{3})$	$\sqrt{3}$	$(\sqrt{3},\infty)$
$f''(x)$	-ve	0	+ve	0	-ve	0	+ve
concavity	down		up		down		up.

<u>Interval of increase/decrease</u>:

$$f'(x)=0$$

$$\Rightarrow (1-x^2)\,e^{-x^2/2}=0 \Rightarrow x=-1,1$$

<u>Note</u>: $f'(-2)=(1-2^2)\,e^{-2}<0$, $f'(0)=1>0$, $f'(2)<0$

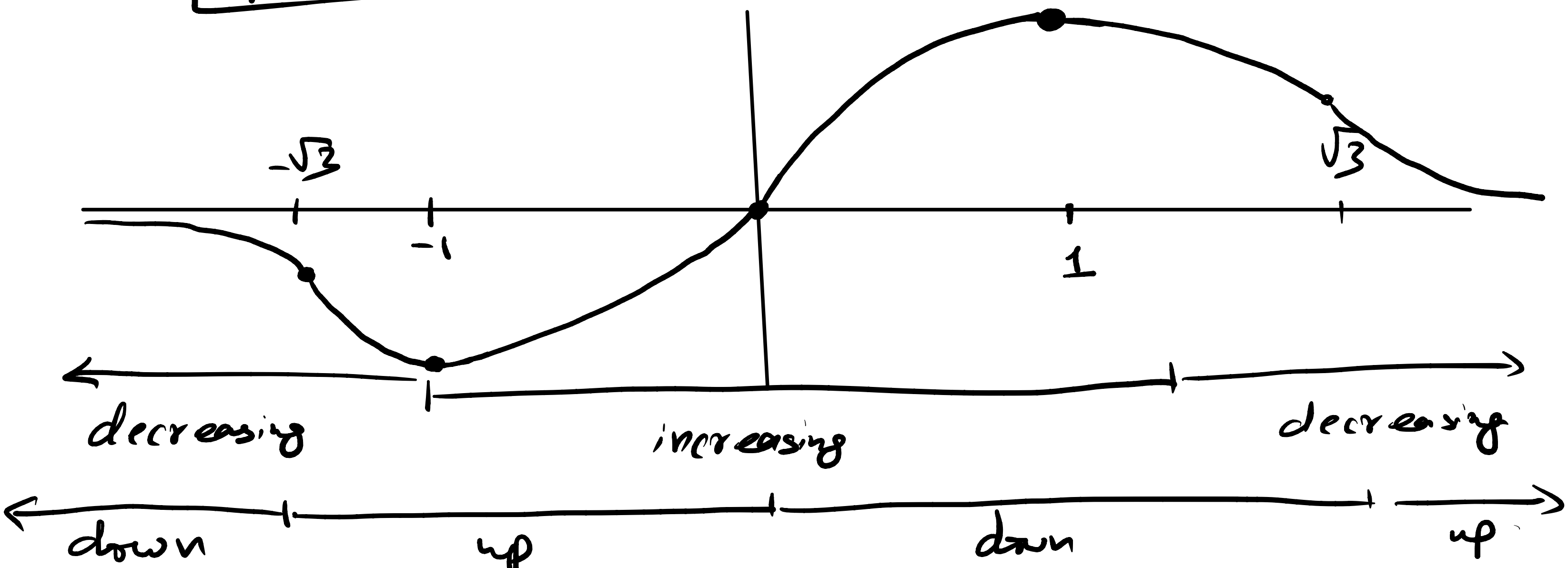
interval	$(-\infty,-1)$	-1	$(-1,1)$	1	$(1,\infty)$
$f'(x)$	-ve	0	+ve	0	-ve
$f(x)$	decreasing		increasing		decreasing.

Where is $f(x)$ positive / negative?

Sign $f(x)$ can change at x-intercepts.

$$f(x)=0 \Rightarrow x e^{-x^2/2}=0 \Rightarrow x=0$$

interval	$(-\infty, 0)$	0	$(0, \infty)$
$f(x)$	$-$ve	0	$+$ve.



Second derivative test:

Let $y=f(x)$ be a function and suppose $x=a$ is a critical point. $(f'(a)=0)$. The second derivative test says:

- If $f''(a)>0$, then $x=a$ is a local min
- If $f''(a)<0$, then $x=a$ is a local max.
- If $f''(a)=0$, then test is inconclusive.

Eg: $f(x)=x^3$, $x=0$ is a critical point and $f''(0)=0$.

Eg: $f(x)=x^3+2x^2-15x$. What does 2nd derivative test say about critical point $x=-3$.

$f'(x)=3x^2+4x-15$, $f''(x)=6x+4$, $f''(-3)<0$