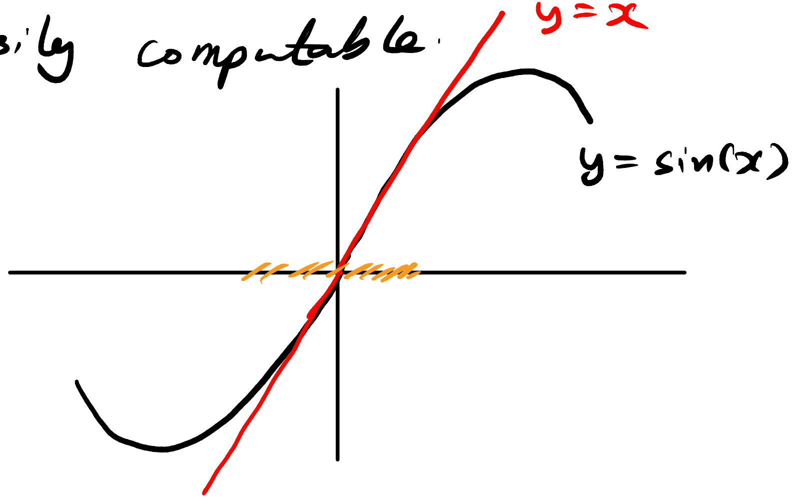
Approximating functions near a specified point

Using a calculator, we can compute

$$\sin\left(\frac{1}{10}\right) \approx 0.09983341\ldots$$

How does a calculator compute this?

The idea is to approximate $\sin(x)$ using functions which are easily computable.



We see the graphs of $y = \sin(x)$ and $y = x$ are very close near $x = 0$

Hence, we can compute $f(x) = \sin(x)$ with $F(x) = x$ near $x = 0$.

$$f(0.1) \approx F(0.1)$$
$$\Rightarrow \sin(0.1) \approx 0.1$$

The error in the approximation is

$$E(x) = | F(x) - f(x) |$$
$$E(0.1) = | 0.1 - 0.09983341\ldots | \approx \text{small}$$

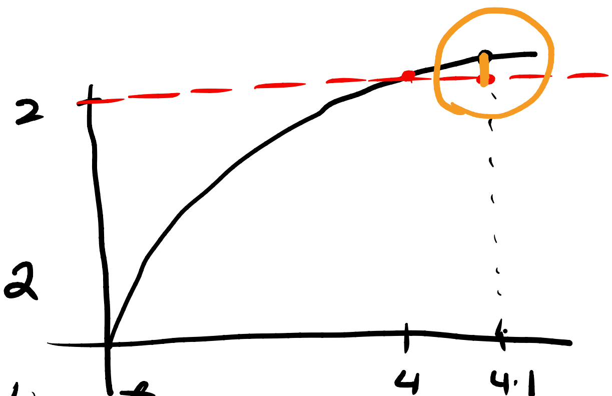
Our goal is to come up with a systematic way to approximate function and to understand the error in the approximation

using polynomials

Example

Let $f(x) = \sqrt{x}$

a) Write down the degree 0 polynomial (or constant) approximation of $f(x)$ at $x=4$.



Requiring: $F(x)$ is degree 0 polynomial

$$F(4) = f(4)$$

The constant approx. at $x=4$ is $F(x) = 2$

b) Use the constant approx. above to estimate $f(4.1)$.

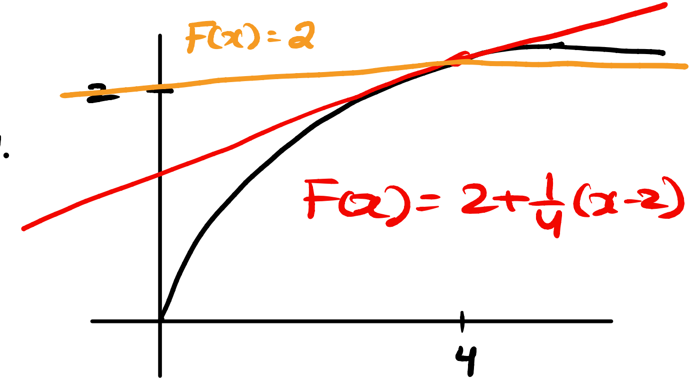
$$f(x) \approx F(x) = 2$$

so, $f(4.1) = \sqrt{4.1} \approx 2.$

$= 2.02485\ldots$

c) Write down a linear approximation of $f(x)$ at $x=4$.

The linear approximation is the tangent line to $y=f(x)$ at $x=4$.



Let $F(x)$ be the linear approx.

Require:
$$F(4) = f(4)$$
$$F'(4) = f'(4)$$

It has a slope of $f'(4)$ and passes through $(4, f(4))$

$$F(x) - f(4) = f'(4)(x-4)$$
$$\Rightarrow \quad F(x) = f(4) + \underline{f'(4)}\,(x-4).$$

Now, $f'(x) = \dfrac{1}{2\sqrt{x}}$, $\quad f'(4) = \dfrac{1}{4}$

So, linear approx, $\boxed{F(x) = 2 + \frac{1}{4}(x-4)}$

(d) Use linear approximation to estimate $f(4.1)$.

$$F(x) = 2 + \frac{1}{4}(x-4)$$

$$f(4.1) \approx F(4.1)$$

$$= 2 + \frac{1}{4}(4.1-4)$$

$$= 2 + \frac{0.1}{4} = \underbrace{2.025}$$

Note that $\sqrt{4.1} = 2.02485\ldots$ So, our approximation is close.

Error, $E(4.1) = |F(4.1) - f(4.1)|$

$$= |2.025 - 2.02485\ldots|$$

$$= 0.00015\ldots$$

Let $f(x)$ be a function which we wat to approximate near $x=a$ by a function $F(x)$.

Constant approx (degree 0)

Require $\left.F(a)=f(a)\right\} \longrightarrow F(x)=f(a)$

Linear approx. (degree 1)

Require $\left.\begin{array}{l} F(a)=f(a) \\ F'(a)=f'(a) \end{array}\right\} \longrightarrow F(x)=f(a)+f'(a)(x-a)$

Quadratic approximation (degree 2).

Require: $F(a)=f(a)$, $F'(a)=f'(a)$ & $F''(a)=f''(a)$.

Find a quadratic polynomial $F(x)$ satisfying these conditions

Quadratic approximation

7

Any quadratic function con written as:

$$F(x) = A + B(x-a) + C(x-a)^2$$

Want to find A, B, and C. Note that

$$F'(x) = B + 2C(x-a), \quad F''(x) = 2C$$

Since function value, 1^{st} derivative, 2^{nd} derivative are equal:

$$F(a) = f(a) \Rightarrow F(a) = A = f(a).$$

$$F'(a) = f'(a) \Rightarrow B = f'(a)$$

$$F''(a) = f''(a) \Rightarrow \boxed{2C = f''(a)} \Rightarrow C = \frac{1}{2} f''(a).$$

Hence, the quadratic approx. to $f(x)$ at $x=a$ is:

$$\boxed{F(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2}$$

Example

Find the quadratic approx. to $f(x)=\sqrt{x}$ at $x=4$.
and use it to approx. $f(\underline{4.1})$.

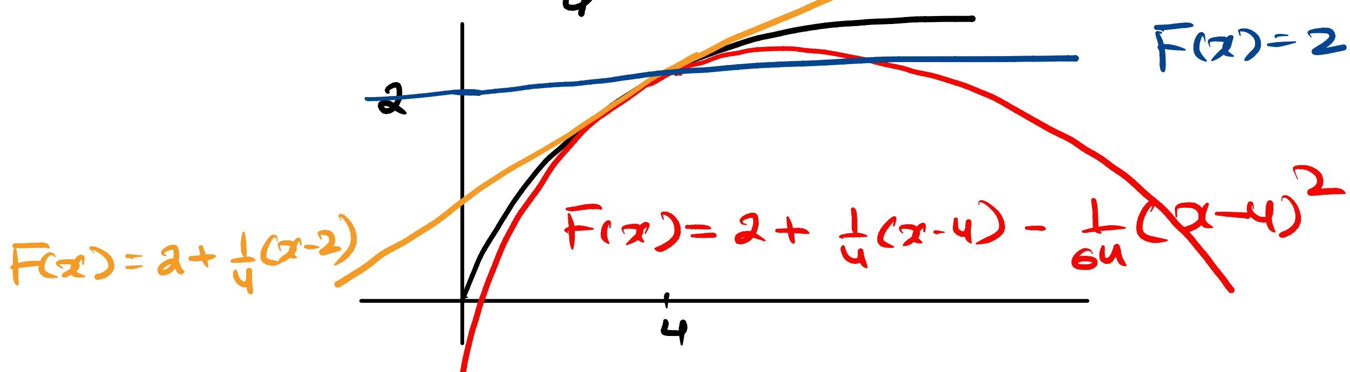
Solution: Here $a=4$ in the quadratic approximation.

$$f(x)=x^{1/2}, \quad f'(x)=\frac{1}{2}x^{-1/2}, \quad f''(x)=-\frac{1}{4}x^{-3/2}$$

$$f(a)=2, \quad f'(a)=\frac{1}{4}, \quad f''(a)=-\frac{1}{4}\cdot\frac{1}{2^3}=-\frac{1}{32}$$

So, the quadratic approx $F(x)$ is

$$F(x)=2+\frac{1}{4}(x-4)+\overset{-1/64}{\frac{1}{2}\cdot\left(-\frac{1}{32}\right)}(x-4)^2$$



Note that $\underset{f(4)}{f(4.1)} \approx F(4.1)$

$$\Rightarrow \sqrt{4.1} \approx 2 + \frac{1}{4}(4.1-4) - \frac{1}{64}(4.1-4)^2$$

$$= 2.02484375\ldots$$

So, error, $E(4.1) = \left|\sqrt{4.1} - 2.02484375\ldots\right|$

$$= 0.000000192$$

Not bad!

<u>Example</u>: Use a quadratic approximation to estimate $e^{0.1}$

$$F(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2$$

We need to choose a point $x = a$, where we can easily evaluate $f(a)$, $f'(a)$, $f''(a)$ and near $x = 0.1$

Choose $a = 0$.

Note: $f(x) = e^x$, $f'(x) = e^x$, $f''(x) = e^x$.

$f(0) = 1$, $f'(0) = 1$, $f''(0) = 1$.

So, $F(x) = 1 + 1\cdot(x-0) + \frac{1}{2}\cdot 1\,(x-0)^2$

$= 1 + x + \frac{x^2}{2}$

Note: $f(x) \approx F(x)$

$\Rightarrow f(0.1) = e^{0.1} \approx 1 + 0.1 + \frac{0.1^2}{2}$

$= 1.105$

True value of $e^{0.1} = 1.10517\ldots$

Taylor polynomial

In general we can approximate a function $f(x)$ at $x=a$ with a degree n polynomial for any $n \geq 0$.

We write $T_n(x)$ for the n^{th} degree Taylor polynomial approximation of $f(x)$ at $x=a$.

$$T_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2$$

$$T_3(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + \frac{1}{3\times 2} f^{(3)}(a)(x-a)^3$$

The notation $f^{(3)}(a)$ means the 3^{rd} derivative of $f(x)$ evaluated $x=a$.

In general,

$$T_n(x) = \frac{1}{0!} f(a) + \frac{1}{1!} f^{(1)}(a)(x-a) + \frac{1}{2!} f^{(2)}(a)(x-a)^2 + \cdots$$
$$+ \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

Here, $n! = 1 \times 2 \times 3 \times \cdots \times n$ "n-factorial"

$\boxed{0! = 1}$ $(n-1)! = \frac{n!}{n}$

The n^{th} degree Taylor polynomial $T_n(x)$ of $f(x)$ at $x=a$ has the property:

$T_n(a) = f(a)$ (also written as $T_n^{(0)}(a) = f^{(0)}(a)$)

$T_n^{(1)}(a) = f^{(1)}(a)$

$\vdots$

$T_n^{(n)}(a) = f^{(n)}(a)$ n^{th} derivative of f.

Summation Notation

$$T_n(x) = f(a) + \frac{1}{1!} f^{(1)}(a)(x-a) + \frac{1}{2!} f^{(2)}(a)(x-a)^2 + \dots + \frac{1}{n!} f^{(n)}(a)\underline{\quad}$$

$(x-a)^n$

This is a bit tedious to write down. Another example.

$$S = 1^2 + 2^2 + \dots + n^2$$

We can use Σ (capital sigma) notation to write this:

$$S = \sum_{k=1}^{n} k^2$$

index $\rightarrow k=1$

This reads as sum of k^2 from k equals 1 to n. Similarly

$$\boxed{T_n(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a)(x-a)^k}$$

The special case $a=0$ is called a Maclaurin polynomial.

Exercise:

Check that $T_4(x)$ has the property $T^{(i)}(a) = f^{(i)}(a)$ for $i = 0, 1, 2, 3, 4$.

$$T_4(x) = \sum_{k=0}^{4} \frac{1}{k!} f^{(k)}(a)(x-a)^k$$

Example: Compute the degree 5 polynomial of e^x at $x=0$. (Compute the degree 5 Maclaurin polynomial of e^x)

Sol^n: Let $f(x) = e^x$. Want $T_5(x)$ at $x=0$.

Note $f(x) = e^x$, $f'(x) = e^x$, ..., $f^{(5)}(x) = e^x$

So, $f(0) = 1$, $f'(0) = 1$, ..., $f^{(5)}(0) = 1$.

$$T_5(x) = f(0) + \frac{1}{1!} f'(0)(x-0) + \dots + \frac{1}{5!} f^{(5)}(0)(x-0)^5$$

$$= 1 + x + \frac{1}{2\times 1} x^2 + \frac{1}{3\times 2\times 1} x^3 + \frac{1}{4\times 3\times 2\times 1} x^4 + \frac{1}{5\times 4\times 3\times 2\times 1} x^5$$

$$T_5(x) = \underbrace{\boxed{1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3}}_{}^{T_3(x)} + \frac{1}{24}x^4 + \frac{1}{120}x^5$$

(b) What is $T_3(x)$?

Note this is just the first 4 terms of $T_5(x)$

So, $T_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$

(c) Use $T_3(x)$ to approximate $e \approx 2.718$

$$e^1 \approx T_3(1) = 1 + 1 + \frac{1}{2}\cdot 1 + \frac{1}{6}\cdot 1 = \frac{6+6+3+1}{6} = \frac{16}{6} = \frac{8}{3}$$

$e^{0.1}$ vs e^1 : $e^{0.1}$ can be approximated more accurately than e^1.

Example

ⓐ Compute the 4th degree Taylor polynomial for $\log(x)$ about $x = 1$

$E(x) = |f(x) - T_n(x)|$

$\leq$ ______ ?

Sol^n Let $f(x) = \log(x)$

want $T_4(x)$ $(a = 1)$

$f(x) = \log(x)$ $\quad f(1) = \log(1) = 0$

$f'(x) = \frac{1}{x}$ $\quad f'(1) = 1$

$f^{(2)}(x) = -\frac{1}{x^2}$ $\quad f^{(2)}(1) = -1$

$f^{(3)}(x) = \frac{2}{x^3}$ $\quad f^{(3)}(1) = 2$

$f^{(4)}(x) = \frac{-6}{x^4}$, $\quad f^{(4)}(1) = -6$

$$T_4(x) = 0 + \frac{1}{1!} \cdot 1 \cdot (x-1) + \frac{1}{2!} \cdot (-1)(x-1)^2 + \frac{1}{3!}(2)(x-1)^3 + \frac{1}{4!}(-6)(x-1)^4$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$$