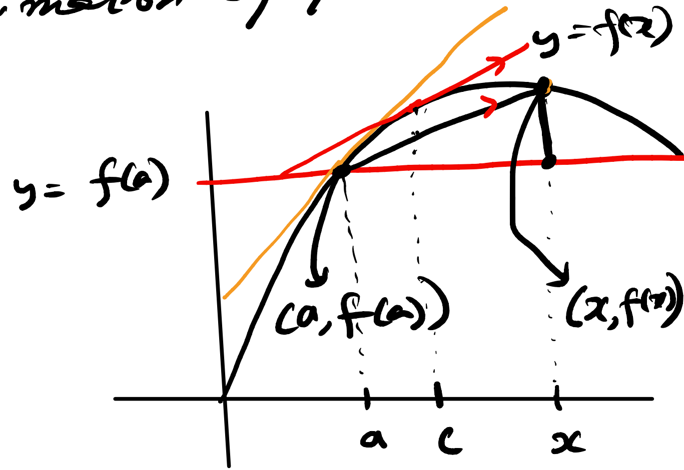
Error in Taylor polynomial approximation

We would like to know the difference

$$\text{Remainder} \rightarrow \underbrace{R_n(x)} = f(x) - T_n(x)$$

between $f(x)$ and the Taylor approximation of $f(x)$ at $x = a$.

Error in constant approx.



$$R_0(x) = f(x) - T_0(x)$$
$$= f(x) - f(a)$$
$$= \frac{f(x) - f(a)}{x-a}(x-a)$$

average rate of change

MVT

$$= f'(c)(x-a), \quad \text{for some } c \text{ between } a \text{ and } x.$$

so, $\boxed{f(x) - T_0(x) = f'(c)(x-a)}$ for some c between a & x.

Example

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Let $f(x) = \sqrt{x}$

a) Use $T_0(x)$ (centered at $x_0 = 4$) to estimate $\sqrt{4.1}$

$$T_0(x) = f(4) = \sqrt{4} = 2$$

$$\Rightarrow f(4.1) \approx T_0(4.1) = 2$$

$$f(x) - T_0(x) = f'(c)(x-a)$$

(b) Using $R_0(x)$ determine if your estimate is an under estimator or over estimator of $\sqrt{4.1}$

$f'(x) = \frac{1}{2\sqrt{x}}$. So, $R_0(x) = f'(c)(x-4) = \frac{1}{2\sqrt{c}}(x-4)$

where $c \in (4, x)$

So, $f(4.1) - T_0(4.1) = f'(c)(4.1-4) = \frac{1}{2\sqrt{c}} \times 0.1 = \frac{0.1}{2\sqrt{c}}$

$$\Rightarrow f(4.1) = T_0(4.1) \underbrace{+\frac{0.1}{2\sqrt{c}}}_{+ve}$$

So, our estimate is an under estimator.

c) Find an upper bound for $|f(4.1) - T_0(4.1)|$

$$f(4.1) - T_0(4.1) = R_0(4.1) = f'(c)(4.1 - 4)$$

$$f'(c)(x-a)$$

$$\Rightarrow |f(4.1) - T_0(4.1)| = \left|\frac{1}{2\sqrt{c}}\, 0.1\right| = \frac{0.1}{2\sqrt{c}} \leq ?$$

where c is between 4 & 4.1.

So, $\frac{1}{2\sqrt{c}} \cdot 0.1 \leq \frac{0.1}{2\sqrt{4}} = \frac{0.1}{4} = 0.025$

Hence, $|f(4.1) - T_0(4.1)| \leq 0.025$

$$R_0(x) = f'(c) \cdot (x-a)$$

$1 \cdot 2 \cdot 3 \cdot 4 = 4!$

Error in Taylor polynomial

Thm Let $T_n(x)$ be the n^{th} degree Taylor polynomial of $f(x)$ at $x=a$.

Lagrange Remainder Thm

$$\underbrace{f(x) - T_n(x)}_{R_n(x)} = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between x and a.

Eg: Let $f(x) = \log(x)$.

Previously, we computed $T_2(x)$ at $x=1$ and determined,

$$f(2) = \log(2) \approx T_2(2) = \frac{1}{2}$$

Find a upper and lower bound for $\underbrace{\log(2) - \frac{1}{2}}_{R_2(2)}$

Want A and B such that

$$A \le \overbrace{f(2) - T_2(2)}^{R_2(2)} \le B$$

$$R_2(2) = \frac{f^{(3)}(c)}{3!}(2-1)^3 \quad \text{for some } c \in (1,2).$$

$$f(x) = \log(x), \quad f'(x) = \frac{1}{x}, \quad f''(x) = -\frac{1}{x^2}, \quad f^{(3)}(x) = \frac{2}{x^3}$$

So, $R_2(2) = \dfrac{2}{6c^3}$ Maximized at $c=1$, Minimized at $c=2$

Hence,

$$\frac{1}{3 \cdot 2^3} \le \underset{\sim}{R_2(2)} \le \frac{1}{3}$$

$$\Rightarrow \quad \frac{1}{24} \le \log(2) - \frac{1}{2} \le \frac{1}{3}$$

$$f(2) - T_2(2)$$

$$\left(T_2(x) = \underset{\text{last class}}{(x-1) + \tfrac{1}{2}(x-1)^2} \right)$$

Example

Compute the 5^{th} degree Maclaurin polynomial of $\sin(x)$.

Maclaurin polynomial is another name for Taylor polynomial at $x = 0$.

Let

$$f(x) = \sin(x)$$
$$f'(x) = \cos(x)$$
$$f^{(2)}(x) = -\sin(x)$$
$$f^{(3)}(x) = -\cos(x)$$
$$f^{(4)}(x) = \sin(x)$$
$$f^{(5)}(x) = \cos(x)$$

$$f(0) = 0$$
$$f'(0) = 1$$
$$f''(0) = 0$$
$$f^{(3)}(0) = -1$$
$$f^{(4)}(0) = 0$$
$$f^{(5)}(0) = 1$$

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

So, $a=0$

$$T_5(x) = \frac{f(0)}{0!} + \frac{f'(0)(x-0)}{1!} + \frac{f''(0)}{2!}(x-0)^2 + \dots + \frac{f^{(5)}(0)}{5!}(x-0)^5$$

$$= 0 + 1(x-0) + 0 + \frac{-1}{3!}(x-0)^3 + 0 + \frac{1}{5!}(x-0)^5$$

$$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$$

$$= x - \frac{1}{6}x^3 + \frac{1}{120}x^5$$

$(T_{11}(x) = ?)$

b) Is your estimate an over/under estimate of $\sin(0.1)$?

Soln: $R_5(x) = f(x) - T_5(x) = \frac{f^{(6)}(c)}{6!}(x-0)^6$

Note $f^{(5)}(x) = \cos(x) \Rightarrow f^{(6)}(x) = -\sin(x)$.

Take $x = 0.1$:

$$\Rightarrow (0.1) - T_5(0.1) = \frac{-\sin(c)}{6!}(0.1)^6$$

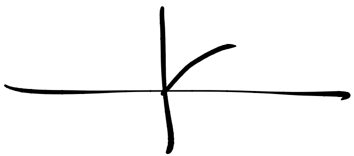
negative

, c is between $(0, 0.1)$.

So, our estimate is overestimate.

(c) Find an M such that $|\sin(0.1) - T_5(0.1)| \le M$.

From $\underline{\underline{b}}$: $|\sin(0.1) - T_5(0.1)| = \left|\frac{-\sin(c)}{6!} \times 0.1^6\right|$, $c \in (0, 0.1)$



$$= \frac{0.1^6}{6!} \cdot \sin(c)$$

So, use $|\sin(c)| \le 1$

$\Rightarrow |\sin(0.1) - T_5(0.1)| \le \frac{0.1^6}{6!} \Rightarrow$ Take $M = \underbrace{\frac{0.1^6}{6!}}$

(This is about 10^{-8}!)

Applications to computing limits

<u>Eg:</u> Compute $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x}$

<u>Sol^n</u>: $\left(\frac{0}{0}\right)$ indeterminat form.

Let $f(x) = e^x - 1$

Taylor polynomial at $x = 0$ (since $x \to 0$ in limit).

$$T_1(x) = f(0) + f'(0)(x-0) = x$$

$$R_1(x) = f(x) - T_1(x) = \frac{f^{(2)}(c)}{2!}(x-0)^2$$

$$\Rightarrow \quad e^x - 1 = x + \frac{f^{(2)}(c)}{2!}(x-0)$$

So, $\lim_{x \to 0} \dfrac{e^x - 1}{x} = \lim_{x \to 0} \dfrac{x + \frac{1}{2!} f^{(2)}(c)\, x^2}{x}$

$= \lim_{x \to 0} \left(1 + \frac{1}{2!} f^{(2)}(c)\, x\right)$

$= 1 \quad \left(x \to 0,\ f^{(2)}(c) \to f^{(2)}(0) \right)$

<u>Compute</u> $\lim_{x \to 0} \dfrac{x^2}{\log(\cos(x))}$

<u>Sol</u>^{n} $\ln(\cos(0)) = \ln(1) = 0$ $\left(\frac{0}{0} \text{ indeterminat form} \right)$

Wait $T_2(x)$ about $x = 0$ ($f(x) = \underline{\ln(\cos x)}$)

$f'(x) = \frac{1}{\cos(x)}(-\sin(x))$ $\quad \underline{f'(0) = 0}$

$= -\tan(x)$

$$f''(x) = -\sec^2(x) \qquad f''(0) = -1$$

So, $T_2(x) = f(0) + f^{(1)}(0)(x-0) + \frac{f^{(2)}(0)}{2!}(x-0)^2$

$$\Rightarrow f(x) = T_2(x) + \frac{f^{(3)}(c)}{3!}(x-0)^3 \qquad \Big|\ f(x) = \ln(\cos x)$$

$$= 0 + 0 + \left(-\frac{1}{2!}\right)x^2 + \frac{f^{(3)}(c)}{3!}x^3$$

$$= -\frac{x^2}{2} + \frac{f^{(3)}(c)}{6}x^3$$

$$\lim_{x\to 0} \frac{x^2 + x}{\ln(\cos(x))} = \lim_{x\to 0} \frac{x^2 / x^2}{\left(-\frac{x^2}{2} + \frac{f^{(3)}(c)}{6}x^3\right)/x^2} = \lim_{x\to 0} \frac{1}{-\frac{1}{2} + \frac{f^{(3)}(c)}{6}\cdot x}$$

$$= -2$$