Logarithmic functions

<u>Defⁿ</u>: Let $q > 1$. Then the logarithm with base q is defined by

$$y = \log_q(x) \iff x = q^y$$

Properties of logarithm

1. $\log_e(1) = 0$ (since e^0 gives 1)
2. $\log_e(e^1) = 1$ (since $\log_e(e^x) = x$)
3. $\log_e(3 \times 5) = \log_e(3) + \log_e(5)$

   $\log_e(a \cdot b) = \log_e(a) + \log_e(b)$ (multiplication becomes addition)
4. $\log_e(2^{10}) = 10 \log_e(2)$

   $\log_e(a^r) = r \log_e(a)$ (bring powers down)

Properties continued.

5. $\log_e(10/3) = \log_e(10) - \log_e(3)$

$\log_e(a/b) = \log_e(a) - \log_e(b)$ provided $b \neq 0$

$\ln(a/b) = \ln(a) - \ln(b)$

Notation: $\underline{\ln}(x) = \underbrace{\log_e}(x)$

2) $\log_e(e^x) = x$

Inverse function

Idea: A function takes an input x and outputs a number y. The inverse function undoes this.

$$f(f^{-1}(x)) = x$$

$$x \longrightarrow \boxed{f} \longrightarrow y \longrightarrow \boxed{f^{-1}} \longrightarrow x$$

$$f^{-1}(f(x)) = x$$

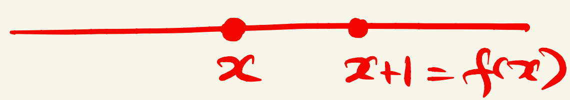
eg: $f(x) = x+1$ has inverse function $f^{-1}(x) = x - 1$

$x = f(x) - 1$

check $y = f(\underset{\substack{\uparrow \\ \text{input}}}{x})$

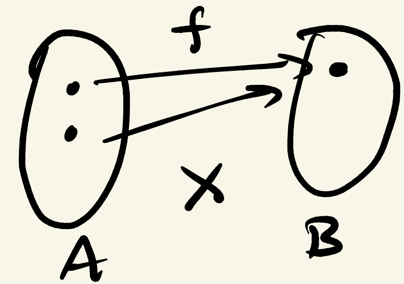
$$f^{-1}(y) = f^{-1}(f(x)) = f^{-1}(x+1) = (x+1) - 1$$
$$= \underset{\substack{\uparrow \\ \text{output of } f^{-1}(y)}}{x}$$

eg: $f(x) = 2x$ has inverse $f^{-1}(x) = x/2$



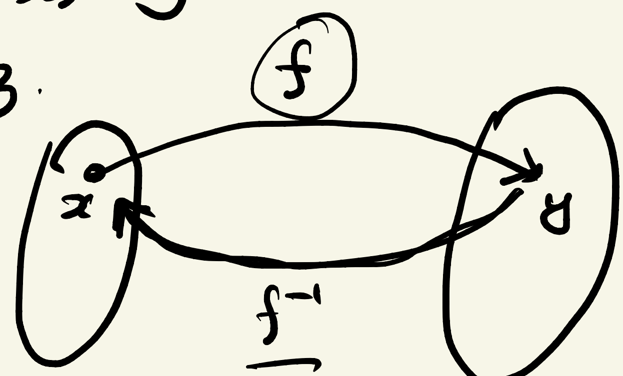
Inverse function

Def^n: Let f be a one-to-one function with domain A and range B. Then its inverse is denoted as f^{-1} and has domain B and range A. It is defined by



$$f^{-1}(x) = y \quad \text{whenever} \quad y = f(x)$$

for any $y \in B$.



So,

- $f^{-1}(f(x)) = x$ for all $x \in A$
- $f(f^{-1}(x)) = x$ for all $x \in B$.

Example

The function $f(x) = \frac{9}{5}x + 32$ converts a temperature x in celsius to fahrenheit.

$$f(100) = \frac{9}{5}x + 32 = 212 \text{ degrees fahrenheit.}$$

Inverse function converts Fahrenheit back to Celsius.

How do we find inverse function?

Let $y = f(x) = \frac{9}{5}x + 32$

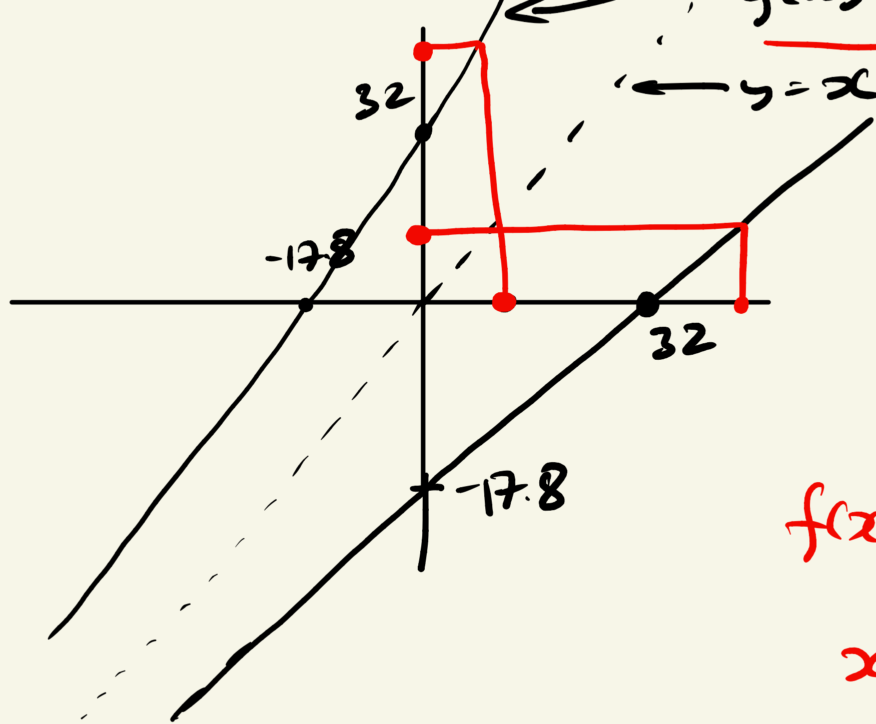
Solve for x in terms of y. So

$$\frac{9}{5}x = y - 32 \Rightarrow x = \frac{5}{9}(y - 32).$$

So, the inverse function is $f^{-1}(x) = \frac{5}{9}(x - 32)$

Reflation over $y=x$ line

$f(x) = \frac{9}{5}x + 32$



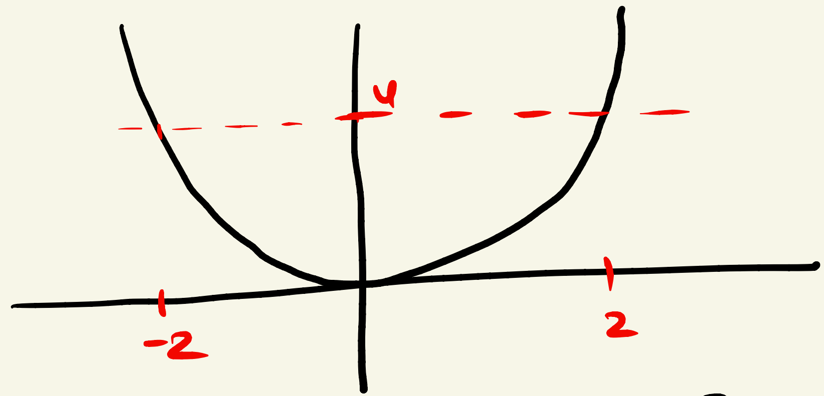
$f(x) = \frac{9}{5}x + 32 = 0$

$x = \frac{-32}{9} \cdot 5$

Inverse function is obtained by reflating the graph with respect to the line $y = x$.

Example

Let $f(x) = x^2$



$f^{-1}(4) = ?$ is it $x = 2$ or $x = -2$?

No unique answer $\Rightarrow$ inverse does not exist.

The horizontal line $y = 4$ intersects the graph at more than one point

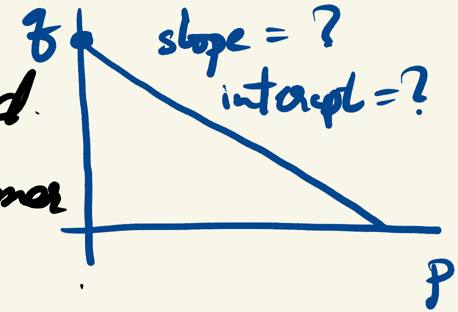
A basic bussiness problem (notes posted on course webpage)

Key terms:
- Linear demand equation
- Total cost
- Revenue
- Break-even point & profit.

Example setting: Opple Inc is the only manufacturer of the popular oPad. Opple estimates that when the price of oPad is \$200, the weekly demand is 5000 units. For a \$1 increase in price, the weekly demand decreases by 50 units. Assume that the fixed cost of production on a weekly basis is \$100,000 and variable cost is \$75 per unit.

Linear demand equation

① Find the linear demand equation for the oPad.



Notation:

- use p for price and q for the weekly demand.
- Demand is the amount q of a good consumer is willing to purchase with the is at p.

we have: $p_0 = 200$, $q_0 = 5000$, and $\Delta p = 1 \Rightarrow \Delta q = -50$.

$y - y_0 = m(x - x_0)$

so, $(q - q_0) = m(p - p_0)$

$\Rightarrow \quad m = q - q_0 / (p - p_0) = -50$

$p_0 = p + 1$

so, the equation of the line is $q - 5000 = -50(p - 200)$

$\Rightarrow q(p) = -50(p - 200) + 5000$

Total cost

b) Find the weekly cost function, $C = C(q)$, for producing q oPads per week.

$$C(q) = \underbrace{\text{fixed cost}}_{\text{doesn't depend on } q} + \underbrace{\text{variable cost}}_{\text{depends on amount of quantity produced}}$$

$$C(q) = 100{,}000 + 75q$$

Note that $C(q)$ is a linear function.

Revenue

c. Find the weekly revenue functions, $R = R(q)$

Revenue is the amount of money the company recieves by selling q goods.

$R(p)$ or $R(q)$

$$R(q) = p \cdot q$$

From a) $\quad q(p) = -50(p-200) + 5000$

$$\Rightarrow p = -\frac{1}{50}(q - 5000) + 200$$

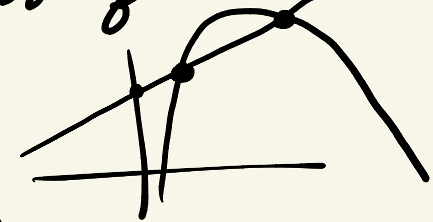
$$= 300 - q/50$$

so, $\quad R(q) = (300 - q/50) \cdot q \rightarrow$ quadratic in q.

$$= 300q - q^2/50$$

Break-even point

d) The break-even points are points where cost equals revenue, i.e. where $C(q) = R(q)$.



$$C(q) = R(q).$$

$$\Rightarrow 100,000 + 75q = q(300 - q/50)$$

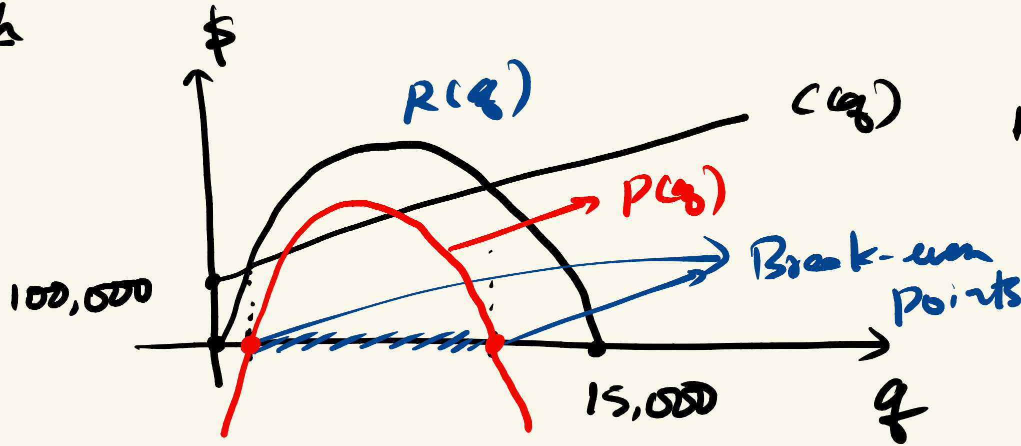
$$\Rightarrow 100,000 + 75q = 300q - q^2/50$$

$$\Rightarrow \frac{1}{50}q^2 - 225q + 100,000 = 0$$

$q \approx 463.5$ or $q \approx 10,786.5$

$C = R \approx \$134.8$ $\quad$ $C = R \approx \$908,984$

Sketch



$C(q) = 100{,}000 + 75q$

$R(q) = \left(300 - \frac{q}{50}\right)q$

$P(q) = -\frac{1}{50}q^2 + 225q - 100{,}000$

Profit ≥ 0 for q in shaded region.

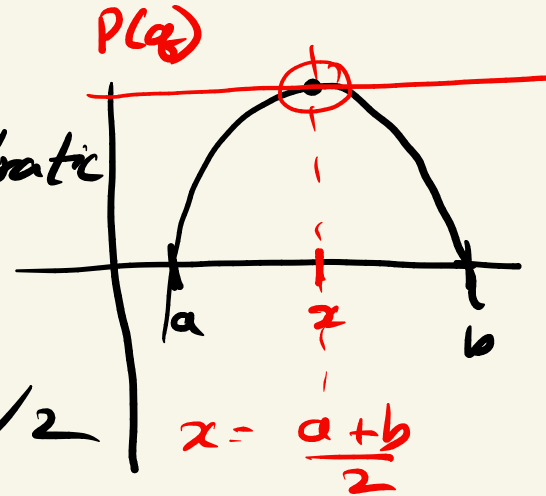
Profit

Profit is defined as: $P(q) = R(q) - C(q)$

so, $P(q) = (300 - q/50)q - (100,000 + 75q)$

$$= -\frac{1}{50}q^2 + 225q - 100,000$$

How should Apple Inc operate in order to maximize the weekly profit $P(q)$?



Need to find the vertex of the quadratic function. The vertex is the average of the two roots.

so, vertex $x = (463.5 + 10,786.5)/2$

$$= 5625$$

& max profit = $\$532,821.5$

Rate of change

Example: You drop a ball from a tall building. Let $s(t)$ be the distance (in meters) the ball falls after t seconds.

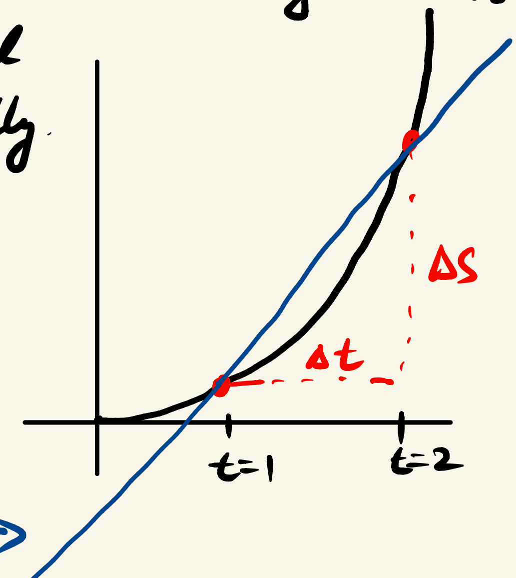
Galileo worked out that $s(t) = 4.9t^2$ $\left(\frac{1}{2} g t^2\right)$

$g = 9.8\,m/s^2$

a) What is the average velocity of the ball between $t=1$ & $t=2$? Interpret graphically.

$$\text{average velocity} := \frac{\text{change in position}}{\text{change in time}}$$

$$= \frac{s(2) - s(1)}{2-1} = 14.7\,m/s$$



slope of the secant line.

b) What is the average velocity between $t=1$ and $t=1+h$? ($h>0$) What happens as h gets closer to 0?

$$\text{average velocity} = \frac{s(1+h)-s(1)}{(1+h)-1}$$

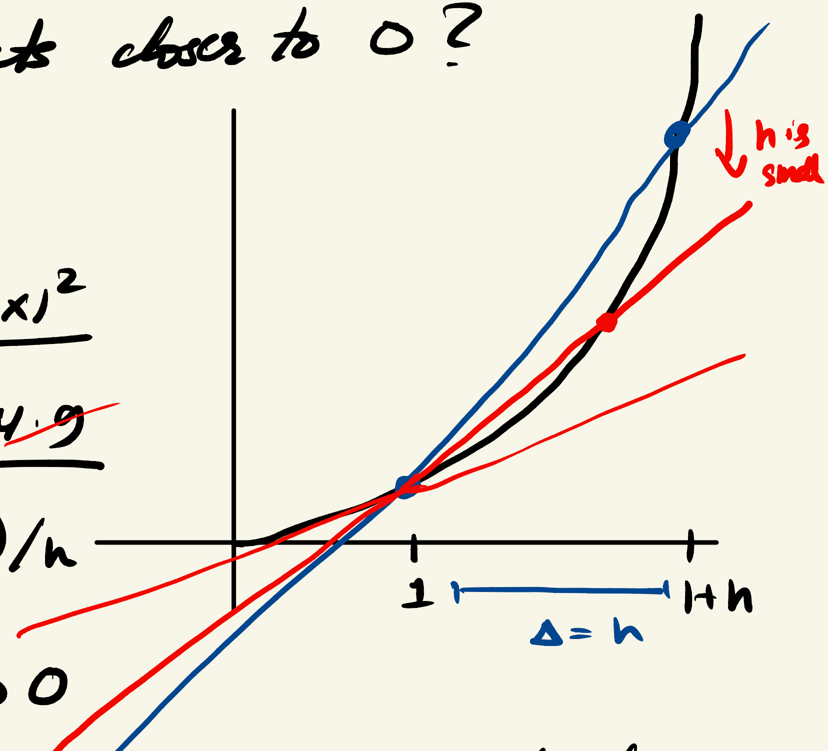
$$= \frac{4.9(1+h)^2 - 4.9\times 1^2}{h}$$

$$= \frac{4.9(\cancel{1}+2h+h^2) - \cancel{4.9}}{h}$$

$$= (9.8h + 4.9h^2)/h$$

$$(h>0) \Longrightarrow \; = 9.8 + 4.9h$$

approaches 9.8 as h gets close to 0



As h gets smaller the secant line gets closer and closer to the tangent line at $x=1$. The slope of the tangent line is the instantaneous velocity of ball at $x=1$.

Limits

Mathematically, we write:

$$\lim_{h \to 0} \frac{s(1+h) - s(1)}{(1+h) - 1} = 9.8$$

to means as h gets closer and closer to 0 (without h being equal to zero) the expression $\frac{s(1+h) - s(1)}{1+h}$ numerically gets closer and closer to $9.8\,m/s$.

Note: we read

$$\lim_{x \to a} f(x) = L \text{ as}$$

limit of $f(x)$ as x approaches a is L.

Example

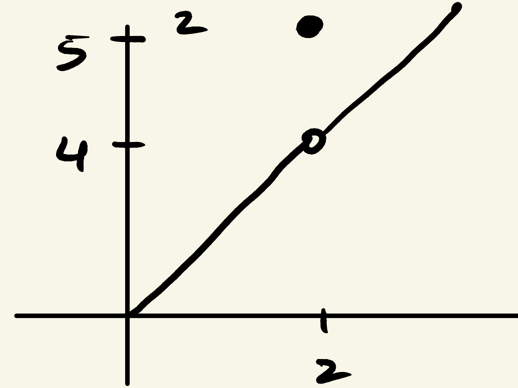
a) $f(x) = \begin{cases} 2x, & x \neq 2 \\ 5, & x = 2 \end{cases}$

$x = 1.9 \Rightarrow f(x) = 3.8$

1.99

⋮

2



What is the $\lim\limits_{x \to 2} f(x)$?

Note that $f(2) = 5$

As x gets closer and closer to 2 without x being equal to 2, $f(x)$ gets closer and closer to 4.

so, $\lim\limits_{x \to 2} f(x) = 4 \neq f(2)$

Example

a) Let $f(x) = \dfrac{x-1}{x^2+3x-4}$ and consider its limit as $x \to 1$

Let $x=1$, $\dfrac{1-1}{1^2+3\times1-4} = \dfrac{0}{0}$ undefined

So, $f(x)$ is undefined at $x=1$

Factorize $x^2+3x-4 = \underline{x+4x-x-4} = (x-1)(x+4)$

so, $\lim\limits_{x\to1} \dfrac{\cancel{x-1}}{\cancel{(x-1)}(x+4)} = \lim\limits_{x\to1} \dfrac{1}{x+4} = \dfrac{1}{5}$

When we encounter $\dfrac{0}{0}$ situations, its often helpful to cancel factors.