Example

Let $f(x) = \frac{x-1}{x^2+3x-4}$ limit of $f(x)$ as x approaches 1.

$$= \frac{\cancel{x-1}}{\cancel{(x-1)}(x+4)} = \frac{1}{x+4}$$

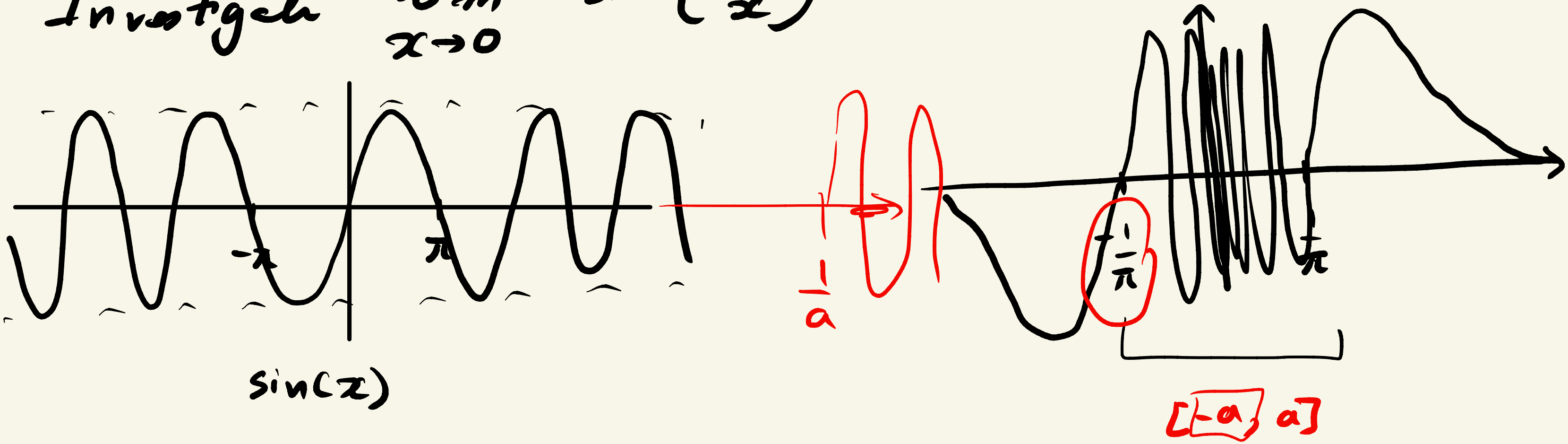
$$\rightarrow \frac{1}{5} \text{ as } x \rightarrow 1$$

$$f(x) \rightarrow \frac{1}{5} \text{ as } x \rightarrow 1 \equiv \lim_{x \to 1} f(x) = \frac{1}{5}$$

Example

Investigate $\lim\limits_{x \to 0} \sin\left(\frac{1}{x}\right)$

$\frac{1}{x} > \pi$

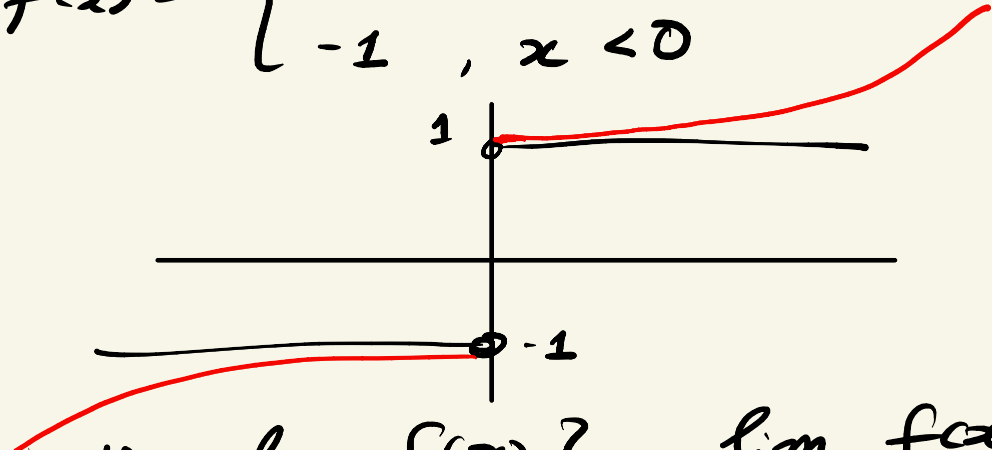


The function does not approach a single value as $x \to 0$. So, the limit does not exist.

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right) = DNE$$

Example

Let $f(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$



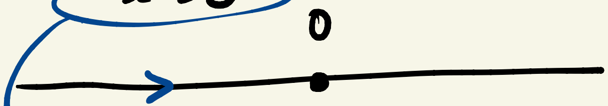
2) What is the $\lim_{x \to 0} f(x)$? $\lim_{x \to 0} f(x) = DNE$

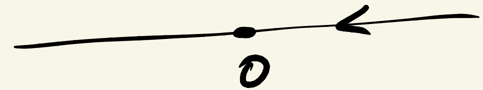
- As x approaches 0 from left of 0, i.e $x < 0$, $f(x)$ gets closer to -1.
- As x approaches 0 from right of 0, i.e $x > 0$, $f(x)$ gets closer to $+1$.

One sided limits

In the previous example, the one sided limit exists:

$$\lim_{x \to 0^-} f(x) = -1 \qquad \lim_{x \to 0^+} f(x) = 1$$





gets closer to zero with $x<0$

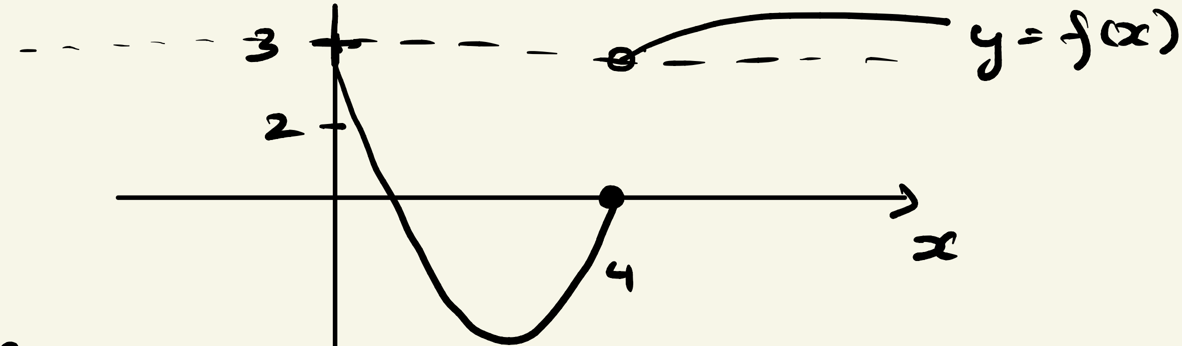
These are called the left and right limits.

two sided limit

Theorem $\lim_{x \to a} f(x) = L$ if and only if

$$\lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = L$$

Example



a) $\lim_{x \to 4^-} f(x) = 0$

b) $\lim_{x \to 4^+} f(x) = 3$

c) $\lim_{x \to 4} f(x) = ?$ (Two-sided limit?)

Since the left and right handed limits are not equal, the limit DNE

Arithmetic of Limits

Thm 1.4.2. in textbook

$$\lim_{x \to a} c = c$$

Eg: Suppose that $\lim_{x\to -1} f(x) = 2$ and $\lim_{x\to -1} g(x) = 3$.

$(2-3)$

Compute:

a) $\lim_{x\to -1} (f(x) \overset{-}{+} g(x)) = 2+3 = 5$

b) $\lim_{x\to -1} (f(x)(g(x))) = 6$

c) $\lim_{x\to -1} (f(x)\cdot c) = 2\cdot c$

set of real numbers

$(c \in \mathbb{R})$

d) $\lim_{x\to -1} f(x)/g(x) = 2/3$

e) $\lim_{x\to -1} (f(x)-2)/(g(x)-3) = ??$

$$f(x) = \frac{x-2}{(x-2)(x-1)}$$

f) $\lim\limits_{x \to -1} (f(x)+g(x))^{1/n} = (2+3)^{1/n} = 5^{1/n}$

$(f(x)/g(x))^{1/n} = \left(2/3\right)^{1/n}$

Key point: We can split limits over multiplication, addition and $(\ )^{1/n}$ provided the limit exists.

Can also split up over division provided limit of denominator is not equal to zero.

Squeeze theorem

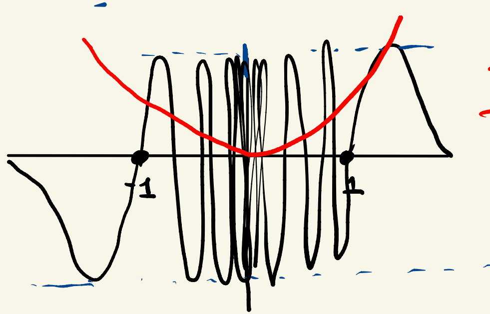
$-100x^2 \le x^2 \sin\left(\frac{\pi}{x}\right) \le 10x^2$

Consider $\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right) = \text{DNE}$

$\lim_{x \to 0} x^2 \sin\left(\frac{\pi}{x}\right) = 0$

$a \cdot b < a$

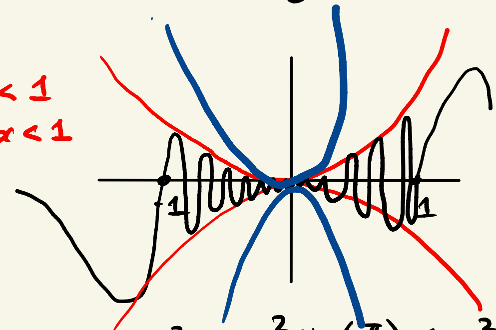
if $b < 1$



$\sin\left(\frac{\pi}{x}\right)$

$x^2 < 1$

for $x < 1$



$-x^2 \le x^2 \sin\left(\frac{\pi}{x}\right) \le x^2$

$-1 \le \sin\left(\frac{\pi}{x}\right) \le 1$

Thm Let $a \in \mathbb{R}$ and f, g, h be function that satisfy

$$f(x) \le g(x) \le h(x).$$

for all x in a interval around a, except for possibly at $x = a$. If $f(x) \to L$ as $x \to a$ and $h(x) \to L$ as $x \to a$, then $g(x) \to L$ as $x \to a$.