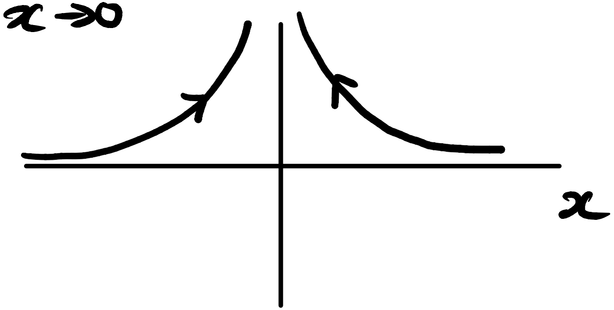
Infinite limits:

Eg: What is $\lim_{x \to 0} \frac{1}{x^2}$?



As $x \to 0$, $\frac{1}{x^2}$ does not approach any real number.

So, the limit does not exists.

However, as $x \to 0$, $\frac{1}{x^2}$ is positive and becomes larger and larger. So, we write

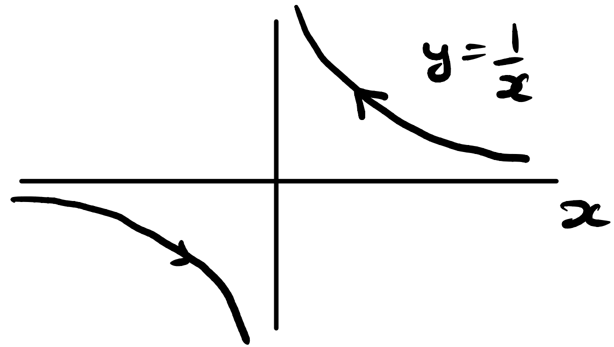
$$\lim_{x \to 0} \frac{1}{x^2} = +\infty$$ ✓ (Note: limit DNE)

DNE ✓

Example

2. $\lim_{x \to 0} -\frac{1}{x^2} = -\infty$

3. $\lim_{x \to 0} \frac{1}{x} = ?$



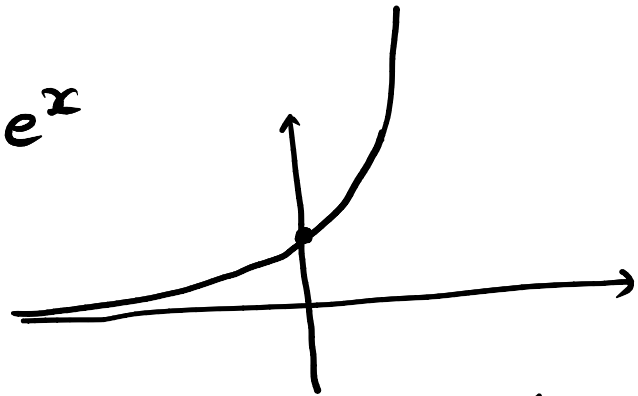
So, $\lim_{x \to 0^-} \frac{1}{x} = -\infty$

$\lim_{x \to 0^+} \frac{1}{x} = +\infty$

but $\lim_{x \to 0} \frac{1}{x} = DNE$. (One sided limits do not agree).

Limit at infinity

Eg: Let $f(x) = e^x$



$e^x = \frac{1}{e^{-x}}$

$e^{10}\uparrow$

What is $\lim_{x \to -\infty} f(x)$? In other words, as x gets large and negative, what value does $f(x)$ approach? $\lim_{x \to -\infty} f(x) = 0$

On the other hand, $\lim_{x \to +\infty} f(x) = +\infty$ because $x \to \infty$ $f(x)$ gets aribitarily large.

Example

$$\lim_{x\to\infty} \frac{x+1}{2x-1}$$

$\frac{x^2+1}{x-1} \to +\infty$ as $x \to +\infty$

$\frac{x-1}{x^2+1} \to 0$ as $x \to +\infty$

Numerator and denominator both go to infinity.

Strategy: Factor out fastest growing term from numerator and denominator.

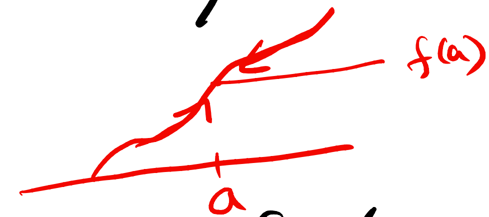
$$\lim_{x\to\infty} \frac{x(1+\frac{1}{x})}{x(2-\frac{1}{x})}$$

So, $$\lim_{x\to\infty} \frac{\frac{x+1}{x}}{\frac{2x-1}{x}} = \lim_{x\to\infty} \frac{1+\frac{1}{x}}{2-\frac{1}{x}}$$

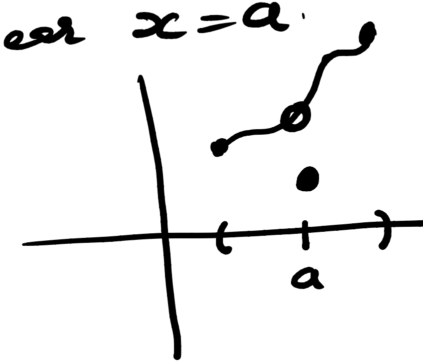
$$= \frac{\lim_{x\to\infty} 1+\frac{1}{x}}{\lim_{x\to\infty} 2-\frac{1}{x}} = \frac{1}{2}$$

Continuity

Defn: A function $f(x)$ is continuous at a point $a \in \mathbb{R}$ if $\lim_{x \to a} f(x) = f(a)$.



Roughly speaking, $f(x)$ is continuous at $a \in \mathbb{R}$ if it doesnot have any abrupt jumps at/near $x = a$.



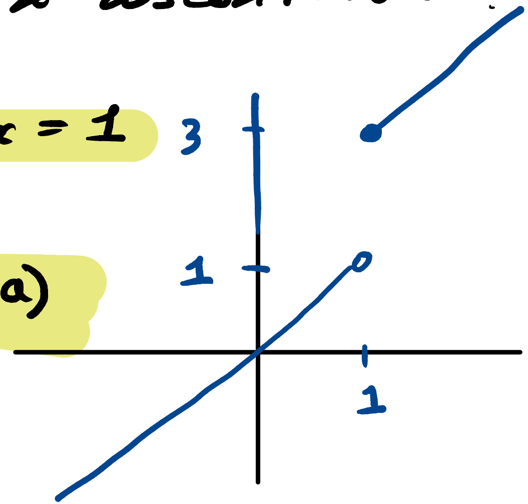
If $f(x)$ is continuous at $x = a$.

- $\lim_{x \to a} f(x)$ and $f(a)$ exists.
- $\lim_{x \to a^-} f(x) = f(a)$
- $\lim_{x \to a^+} f(x) = f(a)$

} one sided limits are equal and is $f(a)$.

Example

Let $f(x) = \begin{cases} x, & x < 1 \\ x+2, & x \geq 1 \end{cases}$. Where is $f(x)$ continuous & discontinuous?



Need to consider $x < 1$, $x > 1$ & $x = 1$

- Suppose $a < 1$

$$\lim_{x \to a} f(x) = \lim_{x \to a} x = a = f(a)$$

So, $f(x)$ is continuous for $x < 1$.

- Suppose $a > 1$

$$\lim_{x \to a} f(x) = \lim_{x \to a} x+2 = a+2 = f(a)$$

So, $f(x)$ is continuous for all $x > 1$.

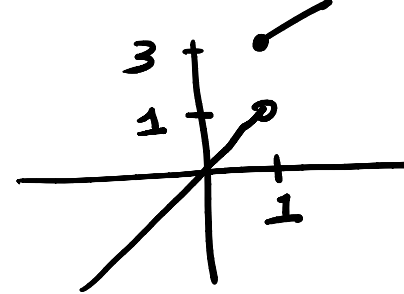
$f: [1, \infty) \to \mathbb{R}$

$f(x) = x+2$

Example 1 contd

Suppose $a = 1$. $\lim\limits_{x \to a} f(x)$

does not exist.



$$\lim_{x \to 1^-} f(x) = 1$$

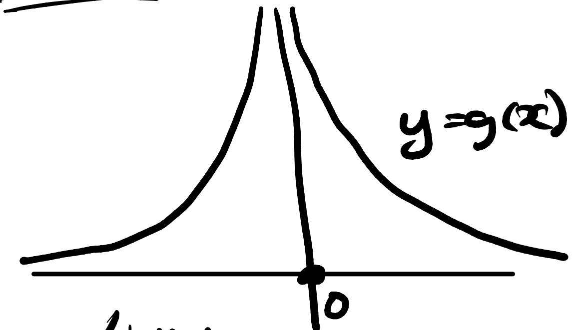
$$\lim_{x \to 1^+} f(x) = 3$$

Since one-sided limit of $f(x)$ at $x = 1$ are not equal, $\lim\limits_{x \to 1} f(x)$ does not exists and $f(x)$ is not continuous at $x = 1$.

Example 2

Let $g(x) = \begin{cases} \frac{1}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

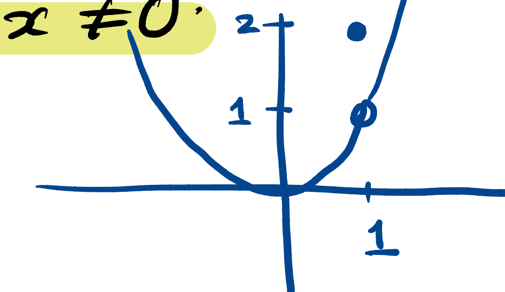
sketch



Where is $g(x)$ continuous?

$\lim_{x \to 0} g(x) \neq g(0) \Rightarrow g(x)$ is discontinuous at $x = 0$.

g(x) is continuous for every point $x \neq 0$.



Zoo of continuous function

Thm The following functions are continuous everywhere in their domain:

1. Polynomials, rational functions (quotient of polynomial)
2. Roots and power
3. Trigonometric functions & their inverse
4. Exponential & logarithms.

We say a function is continuous if it is continuous at every point in its domain.

Example

Where is $f(x) = \dfrac{x^2+x+1}{x-2}$ continuous ?

$x \in \underline{(-2,\infty)} \cup \{-5\}$

domain : $(-\infty, 2) \cup (2, \infty)$

$f(x)$ is a irrational function, so its continuous everywhere it is defined.

Arithmetic of continuity

<u>Thm</u> Suppose $f(x)$ and $g(x)$ are continuous at a point $x=a$. Then the following are also continuous at $x=a$.

① $f(x)+g(x)$

$$\lim_{x\to a}\left(f(x)+g(x)\right)=\lim_{x\to a}f(x)+\lim_{x\to a}g(x)$$

$$=f(a)+g(a)$$

② $f(x)\cdot g(x)$

3) $f(x)/g(x)$ provided $g(a)\neq 0$

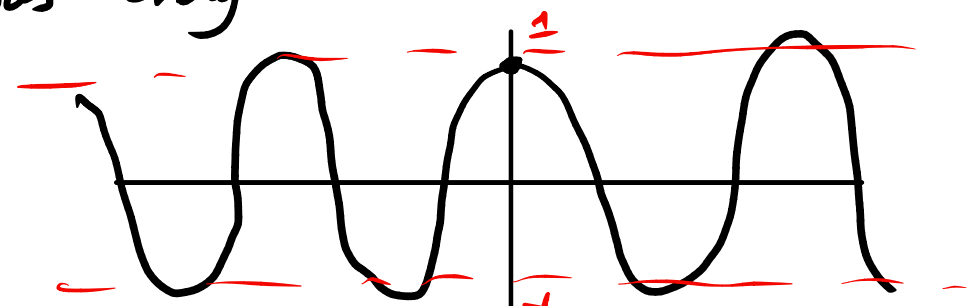
Example

$$f(x) = \frac{\sin(x)}{2+\cos(x)}.$$ Is it continuous?

- Numerator is continuous everywhere
- Denominator: $2+\cos(x)$ is continuous everywhere

& $3 \geq 2+\cos(x) \geq 1$ for all x (because $-1 \leq \cos(x) \leq 1$ for all x)

$\Rightarrow$ $f(x)$ is continuous everywhere.



Example

$f(x) = \underline{\sin}\left(\underline{x^2 + \cos(x)}\right)$ continuous at $x = 0$?

$f(x)$ is a composition of two functions.

$$\left.\begin{aligned} g(x) &= \sin(x) \\ h(x) &= x^2 + \cos(x) \end{aligned}\right\} \Rightarrow f(x) = g(h(x))$$

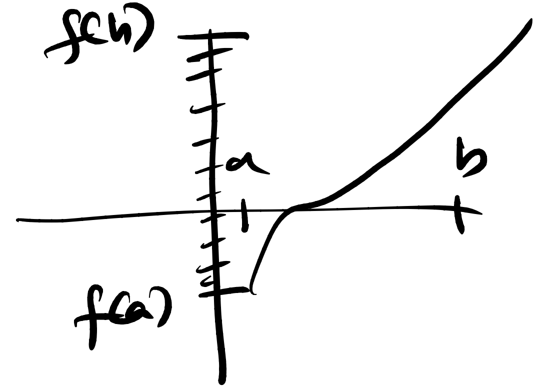
$h(x)$ is continuous everywhere
$g(x)$ is continuous everywhere
$\Rightarrow$ $f(x)$ is continuous everywhere.

composition preserves continuity

Example:

$$f(x) = \frac{x^2+x+1}{x-1}$$

$$g(x) = \frac{x^2+x+1}{x-1} \cdot (x-1)$$

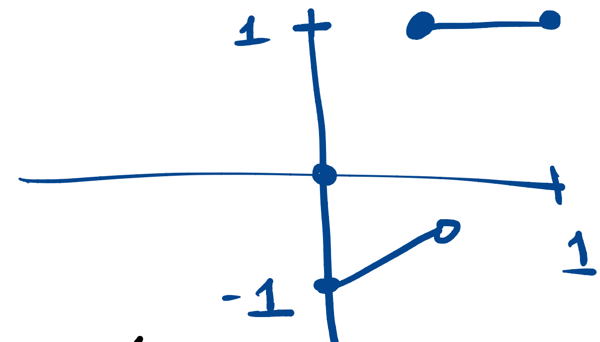


Continuity & intermediate value.

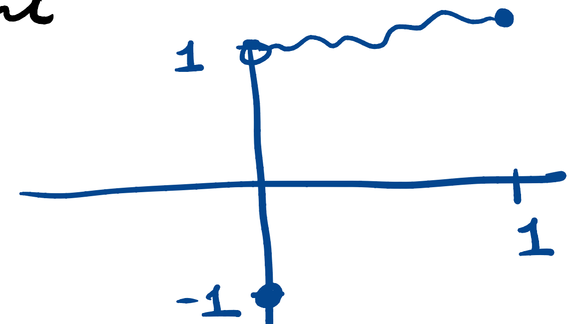
Suppose that $f(x)$ is a function defined on the interval $[0,1]$ and $f(0)=-1$ and $f(1)=1$

(a) Must there exists a point $c \in [0,1]$ $(-1 \le c \le 1)$ with $f(0) = 0$?

No!



(b) What if you are told that $f(x)$ is continuous at every point in $(0,1)$ ← open interval.



No, The function could be right discontinuous.

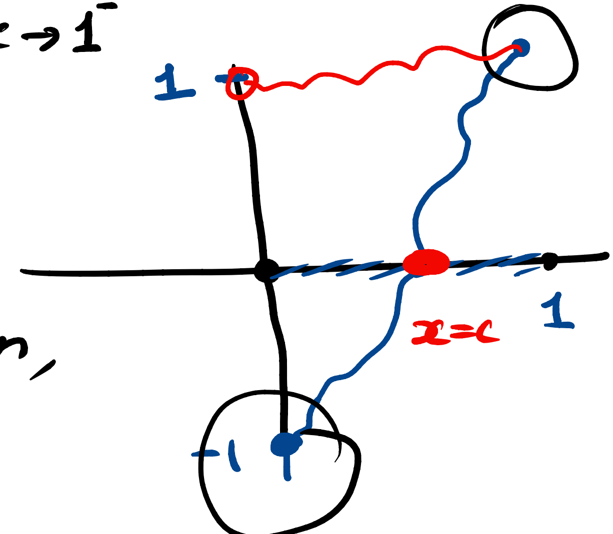
Continuity and intermediate value

c) What if $f(x)$ is continuous at every point in $(0,1)$

& is right continuous at 0 $\left(\lim_{x\to 0^+} f(x) = f(0)\right)$

& is left continuous at 1 $\left(\lim_{x\to 1^-} f(x) = f(1)\right)$

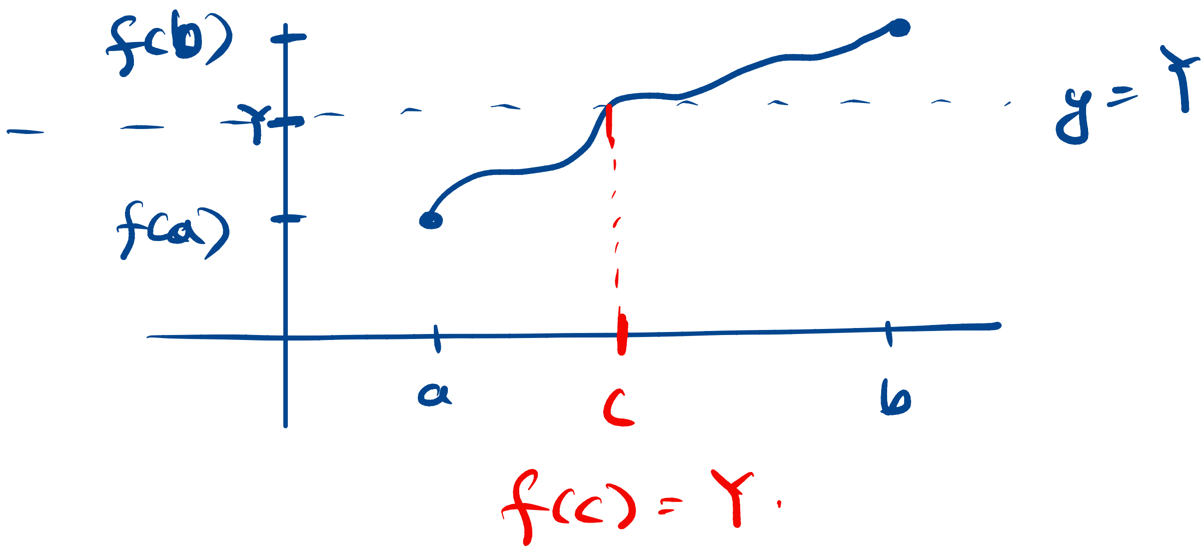
Yes If we trace the graph of the function from $(0,-1)$ to $(1,1)$ without lifting our pen, we must cross x-axis.

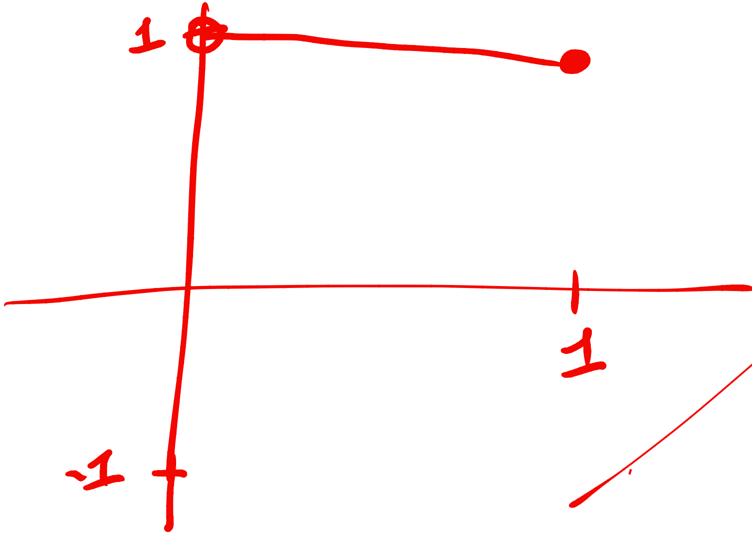


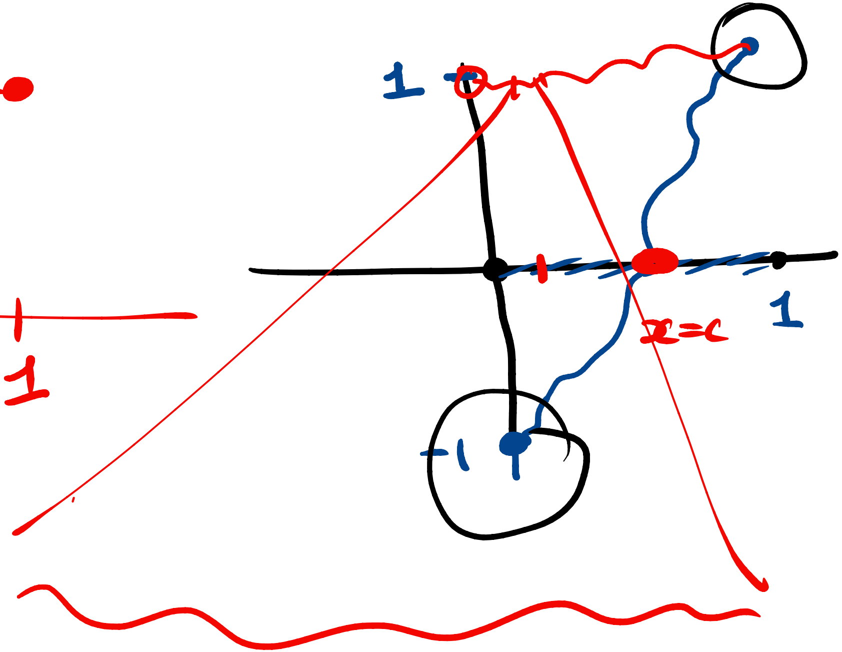
In c) we say the function is continuous on the closed interval $[0,1]$.

Intermediate value theorem.

Let $a<b$ and $f(x)$ be a function that is continuous at all points $a \le x \le b$ ($x \in [a,b]$). If Y is a number between $f(a)$ & $f(b)$, then there exists some $c \in [a,b]$ such that $f(c) = Y$.



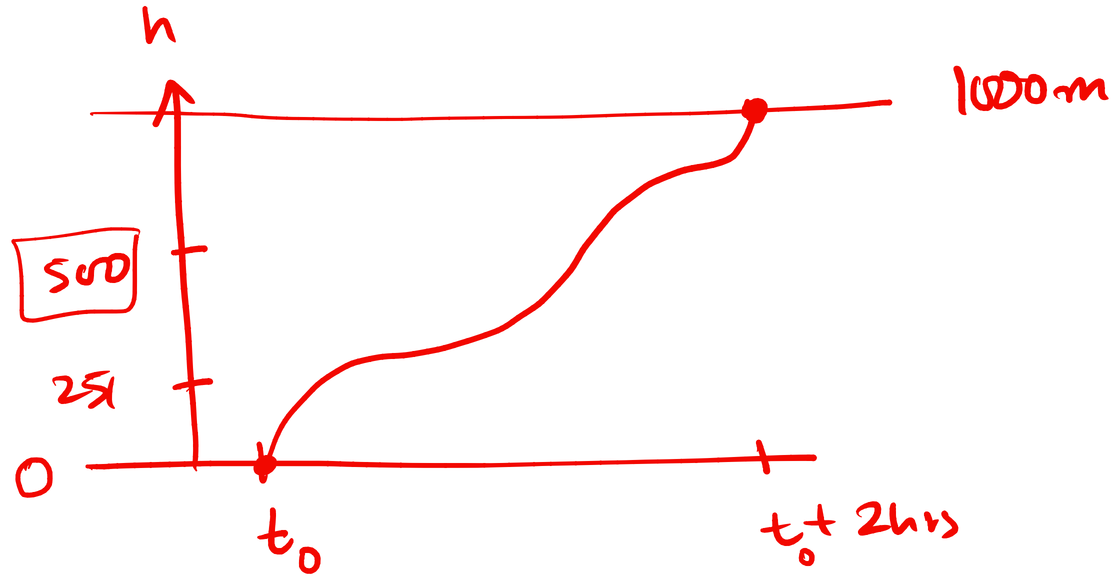




$$\lim_{x \to -\infty} \sin\left(\frac{\pi}{2}\frac{|x|}{x}\right) + \frac{1}{x}$$

$$\sin\left(-\frac{\pi}{2}\right)$$

$$-1$$



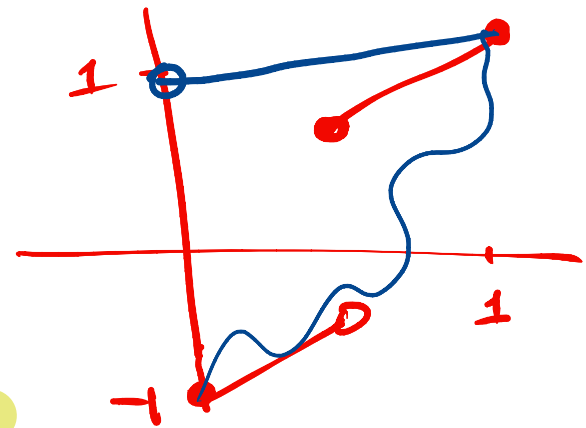
$[t_0, t_0+2]$

between $f(t_0)$ & $f(t_0+2)$.

(a) $f(0) = -1$, $f(1) = 1$ — (1)

Does every f that ~~st~~ satisfy (1) also satisfy the following:

there exist a $c \in [0,1]$ s.t. $f(c) = 0$.



(b) Also assume f is continuous on $(0,1)$.

(c) Also assume f is continuous on $[0,1]$.