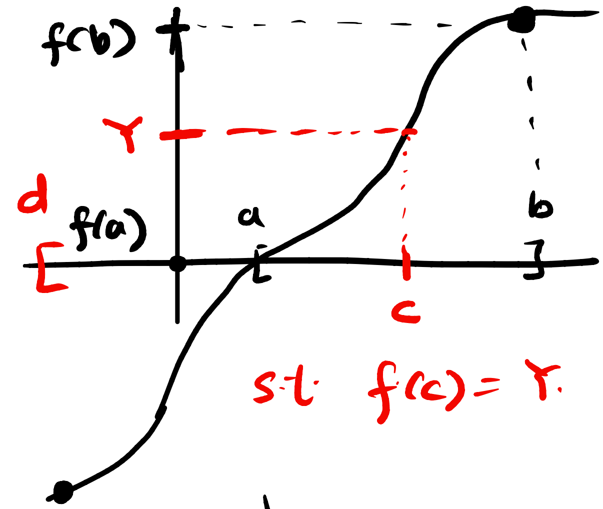
# Intermediate value thm

Show that $\boxed{f(x) = x - 1 + \sin\left(\frac{\pi x}{2}\right)}$ has a zero in $[0,1]$.



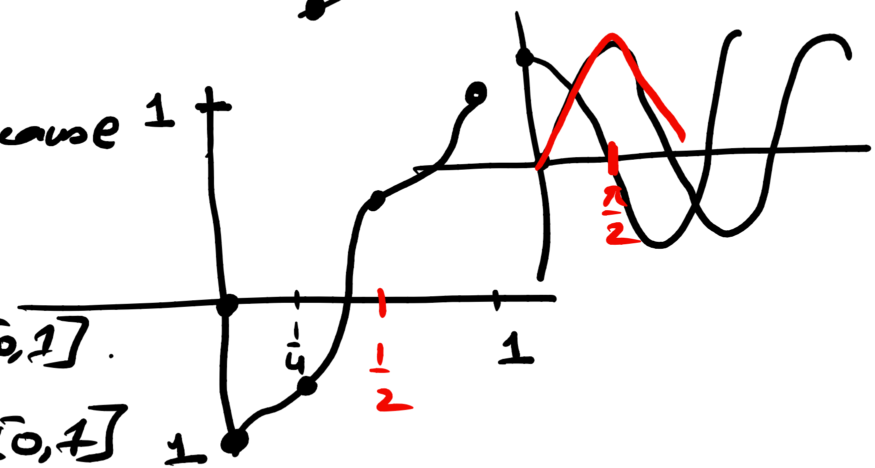

$$f(0) = 0 - 1 + \sin\left(\frac{\pi \cdot 0}{2}\right) = -1$$

$$f(1) = \cancel{1} - \cancel{1} + \sin\left(\frac{\pi \cdot 1}{2}\right) = 1$$

$f$ is continuous on $[0,1]$ because

$x-1$ is continuous on $[0,1]$

& $\sin\left(\frac{\pi x}{2}\right)$ is continuous on $[0,1]$.

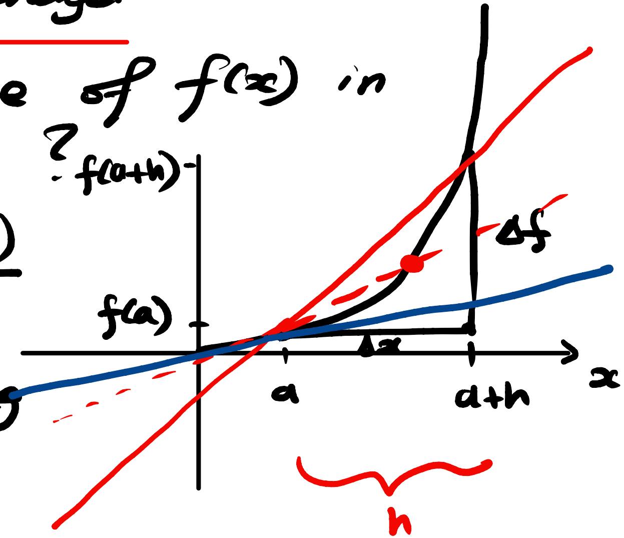



Then, there exists a point $c \in [0,1]$

s.t. $f(c) = 0$

# Average and instantaneous rates of change

What is the average rate of change of $f(x)$ in the interval $[a, a+h]$, $a \in \mathbb{R}$?




$$\text{Average rate of change} = \frac{f(a+h) - f(a)}{h}$$

The slope of the secant line containing $(a, f(a))$, $(a+h, f(a+h))$

Letting $h$ approach zero gives us the instantaneous rate of change of $f$ at $x = a$.

The slope of the tangent line of $f$ at $x = a$.

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

# Derivatives

**Defⁿ:** The derivative of a function $f(x)$ at $x = a$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided the limit exists.

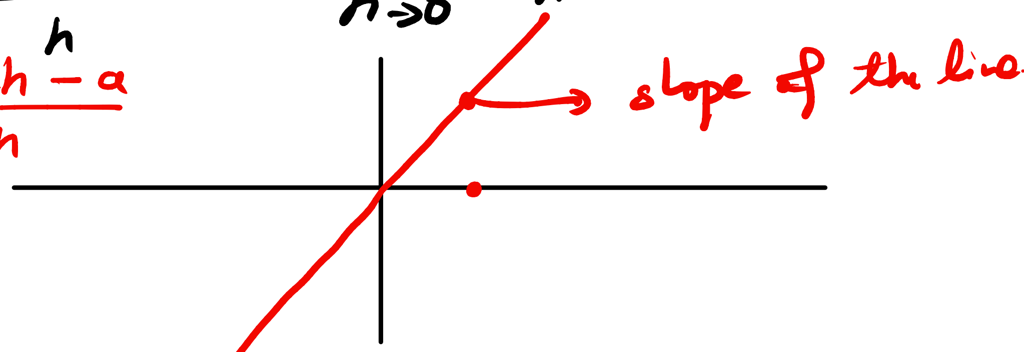
**Eg:** Compute the derivative of $g(x) = x$ at $x = a$

$$g'(a) = \lim_{h \to 0} \frac{g(a+h) - g(a)}{h}$$

$$= \lim_{h \to 0} \frac{(a+h) - a}{h} = \lim_{h \to 0} \frac{h}{h} = 1$$

$$= \lim_{h \to 0} \frac{a+h-a}{h}$$

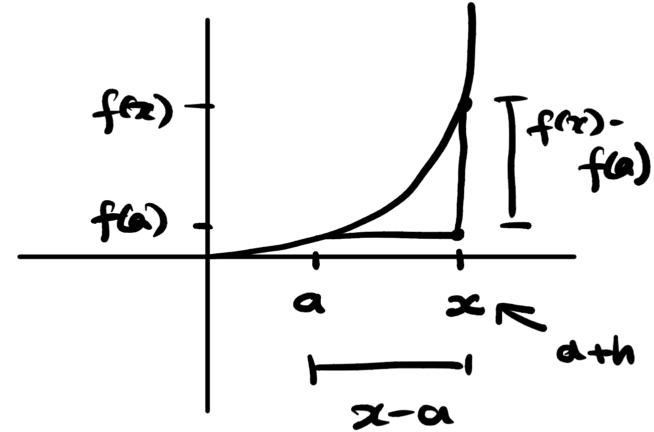
**Visualize:**



# Derivative

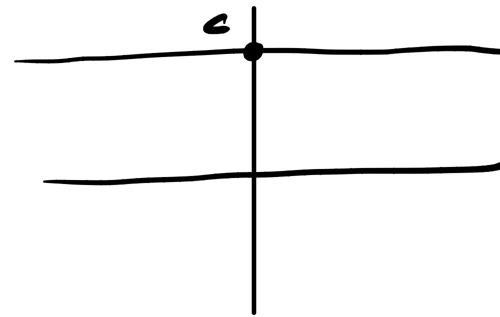
$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided the limit exist.




Simple derivatives.

- Let $f(x) = c$ , $f'(x) = 0$
- Let $g(x) = x$ , $g'(x) = \underline{1}$




Derivative of $f$ at $x = a$

# Derivative as a function

We can also think of derivatives as a function.

**Eg:** For $f(x) = x$, we showed $f'(x)$ at $x = a$ is $1$

That is $f'(a) = 1$

Thus, $f'(x) = 1$

**Def$^n$:** The derivative of $f(x)$ is the function $f'(x)$ with

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \left( = \lim_{y \to x} \frac{f(y) - f(x)}{y - x} \right)$$

If $f'(x)$ exists for all $x$ in $(a, b)$ then we say $f$ is differentiable in $(a, b)$.

## Example

Let $f(x) = \frac{1}{x}$. Use the definition of derivatio to compute $f'(x) = \frac{d}{dx} f(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \leftarrow \text{from definition.}$$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \nearrow \left(\frac{1}{x+h} - \frac{1}{x}\right) \cdot \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x} - (\cancel{x}+h)}{h \cdot x(x+h)}$$

$$= \lim_{h \to 0} \frac{-h}{\cancel{h} \cdot x(x+h)} = \lim_{h \to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}$$

so, $f'(x) = -\frac{1}{x^2}$ provided $x \neq 0$.

# Notation

The following notations are used for "the derivative of $f(x)$ at $x$"

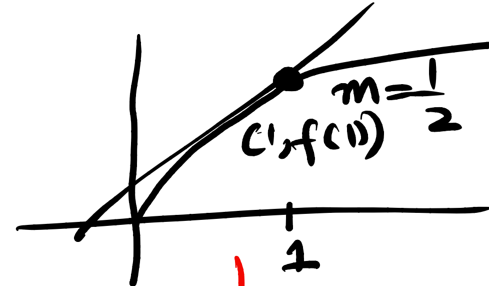
$f'(x)$ $\quad \frac{d}{dx} f(x)$ $\quad \frac{df}{dx}$ $\quad Df(x)$ $\quad D_x f(x)$ $\quad \dot{f}(x)$

These are used for "the derivative of $f(x)$ at $x=a$"

$f'(a)$ $\quad \frac{d}{dx} f(a)$ $\quad \left.\frac{df}{dx}\right|_{x=a}$ $\quad Df(a)$ $\quad D_x f(a)$ $\quad \dot{f}(a)$

## Example

Find the equation of tangent line to $f(x) = \sqrt{x}$ at $x = 1$




slope of tangent line is $f'(1)$

$$f'(1) = \lim_{h \to 0} \frac{\sqrt{1+h} - \sqrt{1}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{1+h} - \sqrt{1}}{h} \frac{(\sqrt{1+h} + \sqrt{1})}{(\sqrt{1+h} + \sqrt{1})}$$

$$\frac{1}{\sqrt{1+h} + 1}$$

$$= \lim_{h \to 0} \frac{(1+h) - 1}{h(\sqrt{1+h} + \sqrt{1})} = \lim_{h \to 0} \boxed{\frac{\cancel{h}}{\cancel{h}(\sqrt{1+h} + \sqrt{1})}} = \frac{1}{2}$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

Also, $(1, f(1)) = (1, 1)$ is a point on the line.

$$\Rightarrow (y - y_0) = m(x - x_0) \Rightarrow y - 1 = \frac{1}{2}(x - 1)$$

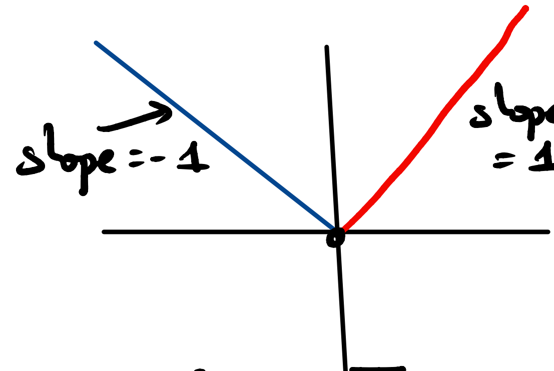
$$\Rightarrow y = \frac{1}{2}x + \frac{1}{2}$$

## Example

Give an example of a function which is continuous at $x=0$ but not differentiable at $x=0$.

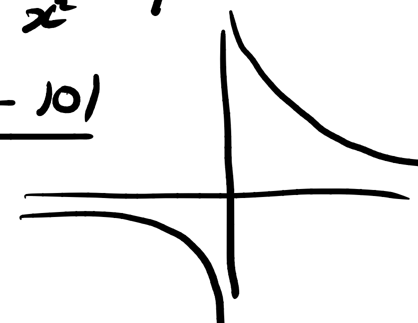
consider $f(x) = |x|$

$$f'(x) = \begin{cases} -1 & \text{if } x<0 \\ \text{DNE} & \text{if } x=0 \\ 1 & \text{if } x>0 \end{cases}$$




$f(x) = \sqrt{x}$

consider $f(x) = \frac{1}{x}$. We showed $f'(x) = -\frac{1}{x^2}$ if $x \neq 0$.



$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0} \frac{|0+h| - |0|}{h}$$

$$= \lim_{h\to 0} \frac{|h|}{h}$$

# Continuity and Differentiability

**Thm** If a function $f(x)$ is differentiable at $x=a$, then $f(x)$ is continuous at $x=a$.

**Idea:**

- $\lim\limits_{x\to a} f(x) = f(a) \iff \lim\limits_{h\to 0} f(a+h) = f(a)$
- $f(a+h) - f(a) = \dfrac{f(a+h)-f(a)}{h} \cdot h$.
- $\lim\limits_{h\to 0}\left(f(a+h)-f(a)\right) = \lim\limits_{h\to 0}\left(\dfrac{f(a+h)-f(a)}{h} \cdot h\right)$

$$= \underbrace{\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}} \cdot \underbrace{\lim_{h\to 0} h}$$

$$= f'(a) \cdot 0 = 0$$

Differentiability $\implies$ continuity

# Rules of differatiation

Our goal is to be able to compute derivative of complicated function by breaking them into simpler derivatives.

**Basic derivatives** $\quad \frac{d}{dx} 1 = 0 \qquad \frac{d}{dx} x = 1$

What is $\frac{d}{dx}(2+3x) = \frac{d}{dx} 2 + \frac{d}{dx} 3 \cdot x = 2 \boxed{\frac{d}{dx} 1} + 3 \boxed{\frac{d}{dx} x}$

$2 \cdot 1$

$= 2 \cdot 0 + 3 \cdot 1$

$= 3.$

- Differentiation splits over addition and substraction
- We can take constants out

# Linearity of differentiation

Let $f(x)$, $g(x)$ be differentiable functions. Let $\alpha, \beta \in \mathbb{R}$ be constants.

For any function $S(x) = \overbrace{\alpha f(x) + \beta g(x)}^{\text{linear combination.}}$

the derivative of $S(x)$ at $x = a$ is

$$S'(x) = \alpha f'(x) + \beta g'(x)$$

Eg: $f(x) = \sqrt{x}$ we "showed" $f'(x) = \frac{1}{2\sqrt{x}}$ for $x \neq 0$

so, $g(x) = \underbrace{5\sqrt{x}}$ then $g'(x) = 5 \cdot \frac{d}{dx}\sqrt{x} = \frac{5}{2\sqrt{x}}$ for $x \neq 0$.

but $h(x) = \sqrt{5x}$ ← this theorem does not apply directly.

$= \underbrace{\sqrt{5}}\,\underbrace{\sqrt{x}}$ ← applies here and $h'(x) = \frac{\sqrt{5}}{2\sqrt{x}}$ $\left(\cancel{\neq \frac{5}{2\sqrt{x}}}\right)$

# Product rule

Let $f(x)$ and $g(x)$ be differentiable function. Then

$$\frac{d}{dx}\left(f(x)\cdot g(x)\right) = \frac{d}{dx}f(x)\cdot g(x) + f(x)\cdot\frac{d}{dx}g(x)$$

$$\frac{d}{dx}\left(h(x)\cdot y(x)\cdot g(x)\right) = f'(x)\cdot g(x) + f(x)\cdot g'(x).$$

**Eg:** Compute $\frac{d}{dx}x^2$. using product rule.

Let $f(x)=x$ , $g(x)=x$

$$\frac{d}{dx}x^2 = f'(x)\cdot g(x) + f(x)\cdot g'(x) = 1\cdot x + x\cdot 1 = \underline{2x}$$

**Eg:** Compute $\frac{d}{dx}x^3$

$$\frac{d}{dx}x^3 = \frac{d}{dx}(x^2\cdot x) = \frac{d}{dx}x^2\cdot x + x^2\cdot\frac{d}{dx}x = 2x\cdot x + x^2\cdot 1$$

$$= 3x^2$$

Eg: $\frac{d}{dx} x^4 = \frac{d}{dx}(x^3 \cdot x)$

$$= \underbrace{\frac{d}{dx} x^3}_{} \cdot x + x^3 \underbrace{\frac{d}{dx} x}_{}$$

$$= 3x^2 \cdot x + x^3 \cdot 1 = 4x^3$$

Can you see the pattern?

$\frac{d}{dx} x^n = n x^{n-1}$ for any $n > 0$, $n$ - integer.

It is more generally true!

## Power rule

• $f(x) = r^x$

$$\frac{d}{dx} x^r = r x^{r-1}$$ for any real number $r$.

Eg: compute $\frac{d}{dx}\sqrt{x} = \frac{d}{dx} x^{\frac{1}{2}} = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$

Using product rule:

• use $\sqrt{x}\cdot\sqrt{x} = x$ to $\frac{d}{dx}\sqrt{x} = x$.

• $f(x) = (2x^2+3x+2)(x^{100}+2)$. What is $f'(-1)$

$$\frac{n^2+5n-n^2}{\sqrt{n^2+5n}+n} = \frac{5n}{\sqrt{n^2+5n}+n} = \frac{\frac{5n}{n}}{\sqrt{\frac{n^2}{n^2}+\frac{5}{n^2}}+1}$$

$$= \frac{5}{\sqrt{1+\frac{5}{n}}+1}$$

$$\frac{(t-8)\cancel{(t+8)}}{2(2t^2+11t-40)}$$

$$2\cancel{(t+8)}(2t-5)$$

$$\frac{t-8}{2(2t-5)}$$

$$(t+8)(2t-5)$$

$$-5t+16t$$

$$=11t$$

$$\frac{\frac{1}{b}-\frac{1}{4}}{b-4} = \frac{?}{\frac{?}{b-4}}$$