Quotient rule

Let $f(x), g(x)$ be differentiable functions.

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)\cdot g(x) - f(x)\,g'(x)}{(g(x))^2}$$

$$\boxed{\frac{d}{dx} f(g(x)) = ?}$$

except at point where $g(x) = 0$

$$\frac{d}{dx}(x^{-1}) = -1x^{-1-1}$$
$$= -x^{-2}$$

Eg: $\frac{d}{dx}\left(\frac{1}{x}\right)$ use quotient rule.

$f(x) = 1$, $g(x) = x$

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{\cancel{f'(x)}^{\,0}\cdot g(x) - f(x)\,g'(x)}{g(x)^2} = \frac{\cancel{0\cdot x} - 1\cdot 1}{x^2} = -\frac{1}{x^2}$$

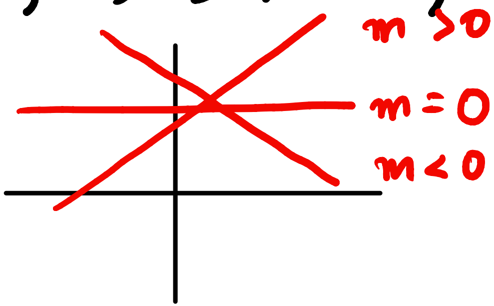
Eg: compute $\frac{d}{dx}\left(\frac{x}{4x+7}\right)$

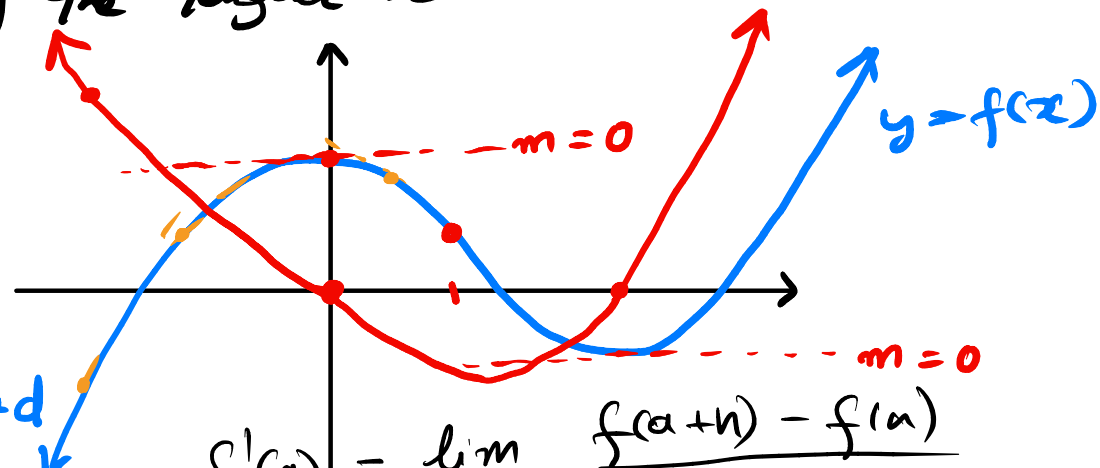
$$\frac{d}{dx}\left(\frac{x}{4x+7}\right) = \frac{1\cdot(4x+7) - x\cdot 4}{(4x+7)^2} = \frac{4x+7-4x}{(4x+7)^2} = \frac{7}{(4x+7)^2}$$

A small detour – sketching the graph of $f'(x)$

Consider $f'(5)$, for some differentiable function $f(x)$.

- $f'(5)$ is the instantaneous rate of change of f at $x=5$
- $f'(5)$ is the slope of the tangent line at $x=5$





$f(x) = ax^3+bx^2+cx+d$

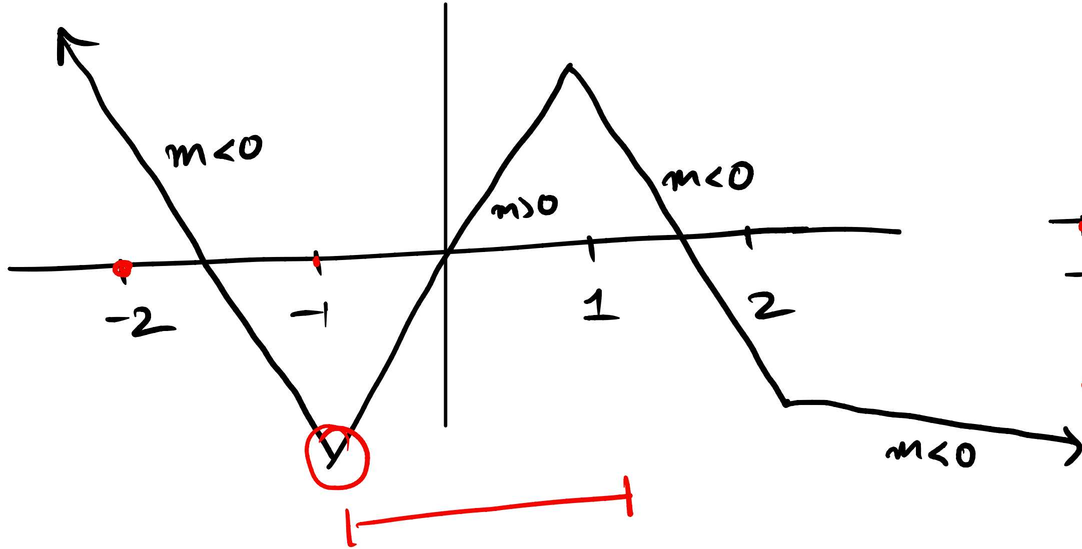
$f(x) = ex^2+fx+g$

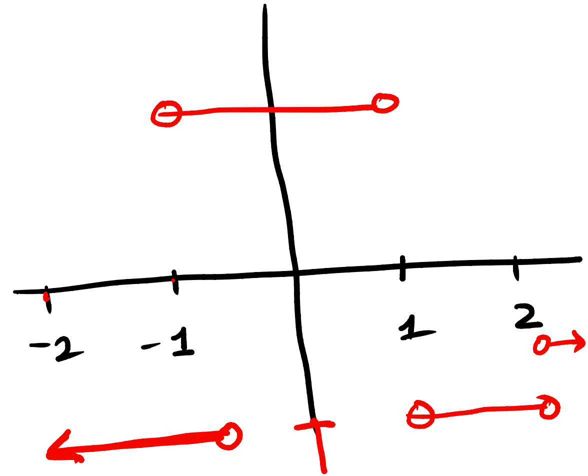
$$f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

+ve

- If $f(x)$ is increasing, $f'(x)$ must be positive
- If $f(x)$ is decreasing, $f'(x)$ must be negative.

Example





$$\frac{f(a+h)-f(a)}{h}$$

Exponential function

Let $a > 0$ and let $f(x) = a^x$, $f(x)$ is an exponential function.

find a s.t. $\frac{d}{dx} a^x = a^x$

Let's try to compute $f'(x)$:

$$\frac{\frac{d}{dx} f(x)}{a^x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

$$\lim_{h \to 0} \frac{a^x(a^h - 1)}{h} = \lim_{h \to 0} \frac{a^x \cdot a^h - a^x}{h} = a^x \boxed{\lim_{h \to 0} \frac{a^h - 1}{h}} \; = 1$$

?

$$= \lim_{h \to 0} a^x \cdot \lim_{h \to 0} \frac{a^h - 1}{h}$$

We cannot evaluate the limit directly, but it exist and is a quantity that only depends on a.

Q: Can we find a value of a so that

$$\lim_{h\to 0} \frac{a^h - 1}{h} = 1 \quad \left(\Rightarrow \frac{d}{dx} a^x = a^x, \text{ for that } a \right)$$

Ans: It turns out we can get this by choosing $a = 2.718\ldots$ This Euler's constant e.

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

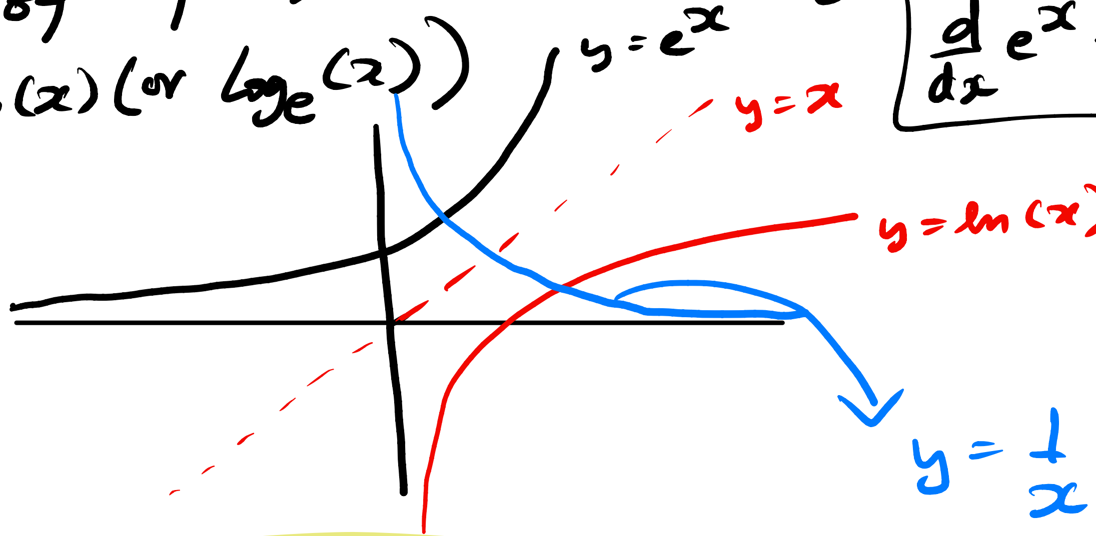
Euler's number is the unique number e such that

$$\frac{d}{dx} e^x = e^x \quad \text{(derivative is equal to itself)}$$

Inverse of e^x

The inverse of $f(x) = e^x$ is the logarithmic function $f^{-1}(x) = \ln(x)$ (or $\log_e(x)$)

$$\boxed{\frac{d}{dx} e^x = e^x}$$



Fact: $\frac{d}{dx} \ln(x) = \frac{1}{x}$

we will prove later, memorize for now.

Example:

1. $h(x) = x \cdot \ln(x)$ $\qquad \frac{d}{dx} h(x) = ?$

2. $\frac{d}{dx}\left[\ln(x e^{x})\right] = ?$

$$f(x) = \frac{x^3+1}{x^2-1}$$

$$1-x^2 = (1-x)(1+x)$$

$$= \frac{(x+1)(\quad)}{(x-1)(x+1)}$$

$$x^2-1 = (x-1)(x+1)$$

$$= \frac{(\quad)}{x-1}$$

$$\lim_{x\to -1} f(x) = \lim_{x\to -1} \frac{(\quad)}{x-1}$$