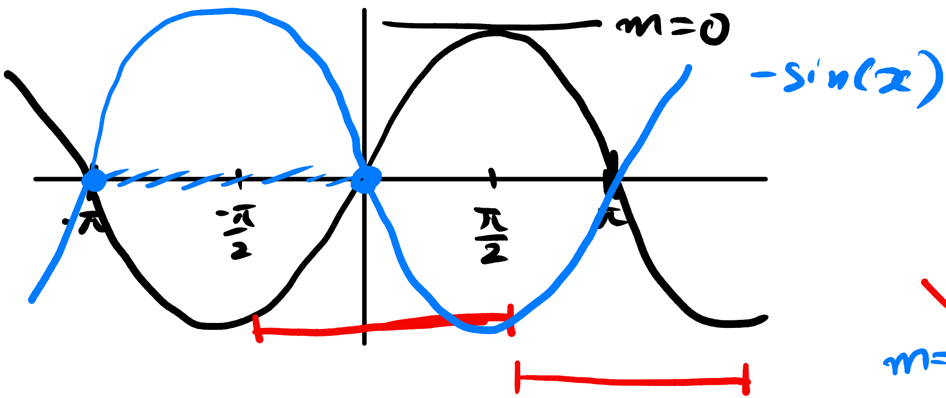
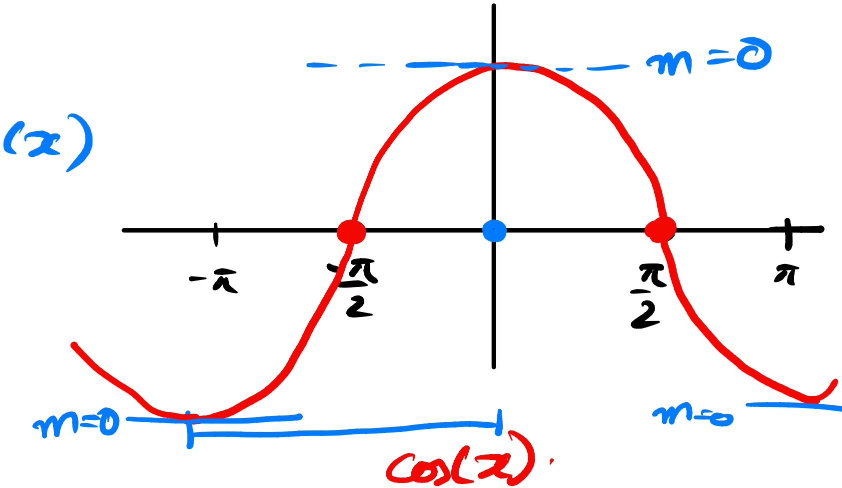
Derivative of trig functions





$\boxed{\sin(x)}$

In fact, $\dfrac{d}{dx}\sin(x) = \cos(x)$

Similarly, $\dfrac{d}{dx}\cos(x) = -\sin(x)$

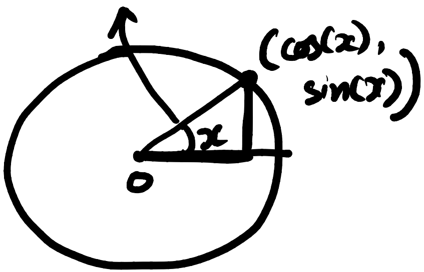
Memorize them!

Example

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

differentiate $f(x) = \tan(x)$

$$\frac{d}{dx}\tan(x) = \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) \leftarrow \text{quotient rule}$$



$$= \frac{\left(\frac{d}{dx}\sin(x)\right)\cos(x) - \sin(x)\frac{d}{dx}\cos(x)}{\cos^2(x)}$$

$$= \frac{\cos(x)\cdot\cos(x) - \sin(x)\cdot(-\sin(x))}{\cos^2(x)}$$

$$= \underbrace{(\cos^2 x + \sin^2 x)}\big/\cos^2 x = 1/\cos^2 x = \sec^2 x$$

Then, $\frac{d}{dx}\tan(x) = \sec^2 x$

$$\frac{d}{dx}\tan(x) = \frac{d}{dx}\left(\sin(x)\cdot\cos^{-1}x\right)$$

Derivative of trig functions

Thm

$$\frac{d}{dx}\sin(x) = \cos(x) \qquad \frac{d}{dx}\cos(x) = -\sin(x)$$

$$\cot(x) = \frac{\cos(x)}{\sin(x)}$$

and consequently:

$$\frac{d}{dx}\tan(x) = \sec^2 x \qquad \frac{d}{dx}\cot x = -\csc^2 x$$

$$\frac{d}{dx}\csc(x) = -\csc(x)\cdot\cot(x) \qquad \frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$

$$\csc(x) = \frac{1}{\sin(x)}$$

$$\tan(x) = \sin(x)\cdot(\cos(x))^{-1}$$

$$(\cos(x))^{-1} = f(g(x)), \quad f(x) = x^{-1}, \quad g(x) = \cos x$$

Chain rule

Lets try to differatiate $f(x) = \sin(x^2+1)$

We cannot differentiate this using what we have learnt so far. Need to use composition of functions.

Let $y = \sin(u)$ and $u = x^2+1$

Note that if we substitute $u = x^2+1$ to $y(u)$ we get $y = \sin(u) = \sin(x^2+1)$ (which is the original function)

We want to compute $\frac{dy}{dx} = \frac{dy}{\cancel{du}} \frac{\cancel{du}}{dx}$ → differentiate

Chain rule: $\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$ $= \cos(u) \cdot \frac{d}{dx}(x^2+1) = \cos(u) \cdot 2x$

$\boxed{= \cos(x^2+1) \cdot 2x}$

Chain rule

Version 1: Let $y = f(u)$ and $u = g(x)$ be differentiable functions. Then

u is a function of x

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

need to substitute u back.

Version 2: We can rewrite the chain rule above.

Substitute $u = g(x)$ in $y = f(u)$ gives us

$$y = \underbrace{f(g(x))}$$

and

$$\frac{d}{dx}\left[f(g(x))\right] = \underbrace{f'(g(x))}_{\text{differentiate outer function}} \cdot \underbrace{g'(x)}_{\text{differentiate inner function.}}$$

Examples:

① Compute $\frac{d}{dx}\tan(x^5)$

dummy variable

outer function: $f(u) = \tan(u)$

inner function: $u(x) = x^5$

$$\frac{d}{dx}\tan(x^5) = f'(u(x)) \cdot u'(x)$$

$$f'(u(x)) = \frac{d}{du}f(u)\Big|_{u=x^5} = \sec^2(u)\Big|_{u=x^5} = \sec^2(x^5)$$

$$u'(x) = 5x^4$$

So, by chain rule: $\frac{d}{dx}\tan(x^5) = \sec^2(x^5)\cdot \underline{\underline{5x^4}}$

2) Find $\frac{d}{dx}(1+3x)^{75}$ $\xrightarrow{}$ linear function.

$f(u) = u^{75}$, $u(x) = 1+3x$

$f'(u) = 75u^{74}$, $u'(x) = 3$

$\Rightarrow \frac{d}{dx} f(u(x)) = 75(1+3x)^{74} \cdot 3 = \underline{225(1+3x)^{74}}$

Thm Let f be differentiable and $g(x) = ax+b$ for some $a, b \in \mathbb{R}$. Then

$g'(x) = a$

$$\boxed{\frac{d}{dx} f(g(x)) = \boxed{a f'(ax+b)} \\ = f'(g(x)) \cdot g'(x)}$$

$= f'(ax+b) \cdot a$ a

3. Find $\frac{d}{dx} e^{\cos(2x)}$

try: $\frac{d}{dx} \ln(\sin(x)) = \cot(x)$. $\frac{d}{dx} (\cos(x))^{-1}$

Lets use $\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$ (remember to substitute back u)

outer function: $f(u) = e^u$

inner function: $u(x) = \cos(2x)$

$\Rightarrow f(u(x))$ gives original function

$$\frac{df}{dx} = \boxed{\frac{df}{du}} \frac{du}{dx}$$ ($u(x)$ is a composition of function)

To compute $\frac{du}{dx}$, $u(v) = \cos(v)$ and $v(x) = 2x \Rightarrow u(v(x)) = \cos(2x)$

So, $\frac{du}{dx} = \frac{du}{dv} \cdot \frac{dv}{dx} = -\sin(v) \cdot 2 = -2\sin(2x)$

$= \frac{df}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$

& $\frac{df}{du} = e^u = e^{\cos(2x)}$

combine

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = e^{\cos(2x)} \cdot (-2\sin(2x))$$

Applications in Economics

Recall the following:

Business want to maximize Profit = Revenue − Cost.

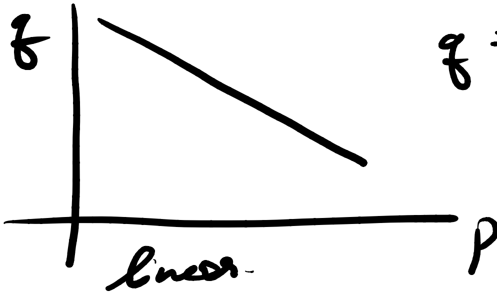
Cost can be modeled as: $\text{Cost} = \underbrace{\text{Fixed cost}}_{\text{independent of } q} + \underbrace{\text{Variable cost}}_{\text{dependent on } q}$

Revenue is: $\text{Revenue} = \text{Price} \times \text{Quantity} = pq$

The demand equation is the equation that relates p and q.

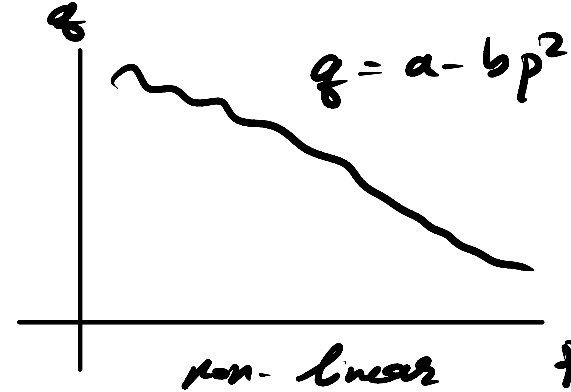
When is profit maximized — related to Marginals.

Demand curve



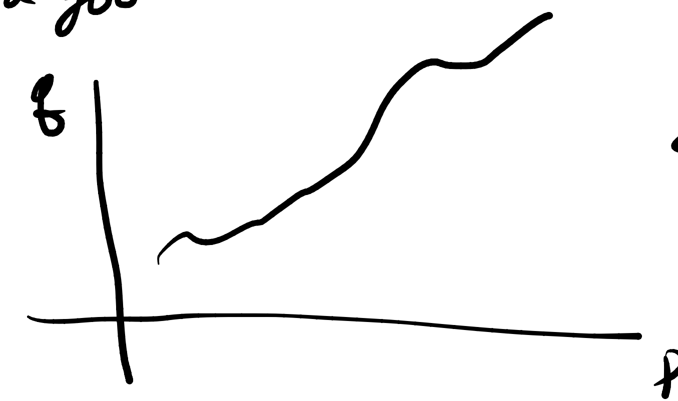
$q = a - bp$

$a, b > 0$



$q = a - bp^2$

The law of demand states that at a higher price, consumer demand of a good is lower. (i.e. as p increase q decreases.



← not generally possible.

Marginal cost

Common economic definition: the additional cost of producing 1 extra unit of good. This is captured by $\underbrace{\frac{\Delta C}{\Delta q}}_{\text{average rate of change}}$

Formal definition

The derivative of cost with respect to quantity q: $MC = \frac{dC}{dq}$ (units are \$/unit).

Example: Cost function $C(q) = \underbrace{2000}_{\text{fixed}} + 10q^2$

$$\text{Marginal cost} = \frac{dC}{dq} = 20q$$

Marginal revenue

Common definition : Additional revenue for 1 extra unit of production.

Formal definition :

Marginal Revenue, $MR = \frac{dR}{dq}$

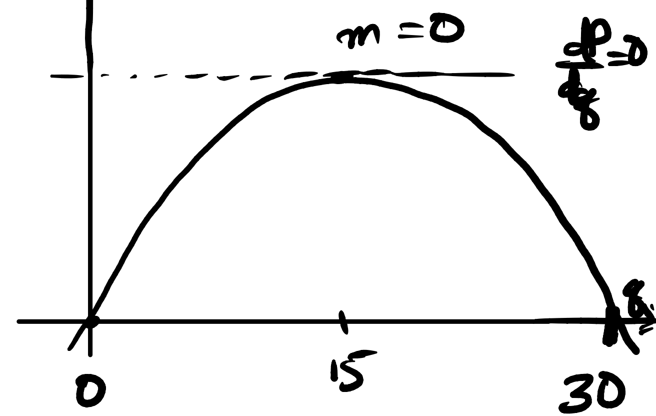
In general: "Marginal $\sim\sim$" $= \frac{d}{dq}(\sim\sim)$

Profit

The profit is $P = R - C$

Eg $P(q) = \underline{600q - 20q^2}$

When is profit maximized?



Profit is maximized when $\boxed{\dfrac{dP}{dq} = 0}$, i.e. marginal profit is zero.

So, $\dfrac{dP}{dq} = 600 - 40q = 0 \Rightarrow q = 15$

In general: $\dfrac{dP}{dq} = \dfrac{dR}{dq} - \dfrac{dC}{dq}$. So, $\dfrac{dP}{dq} = 0 \Rightarrow \boxed{\dfrac{dR}{dq} = \dfrac{dC}{dq}}$

So, profit is maximized when Marginal cost = Marginal revenue.

Example.

Find the optimal output to maximize profit given

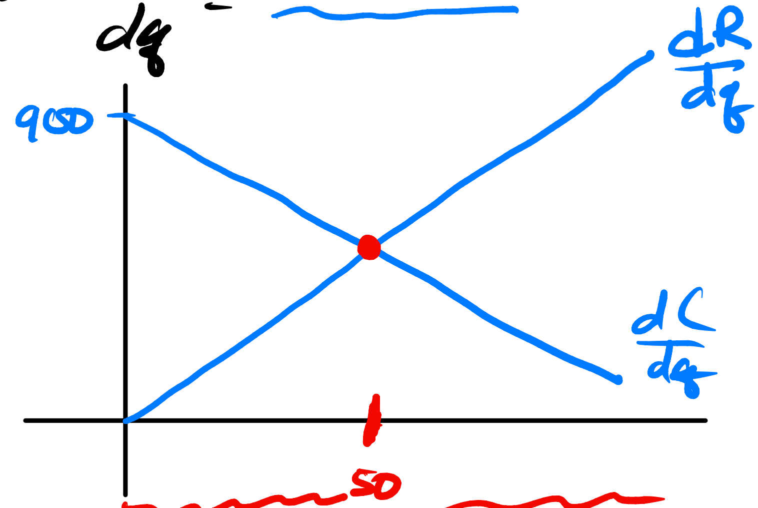
$$\frac{dR}{dq} = 8q \quad \text{and} \quad \frac{dC}{dq} = 900 - 10q$$

Set $\frac{dR}{dq} = \frac{dC}{dq}$

$\Rightarrow 8q = 900 - 10q$

$\Rightarrow 18q = 900$

$\Rightarrow q = 50$



$\frac{dR}{dq} < \frac{dC}{dq}$ $\quad$ $\frac{dR}{dq} > \frac{dC}{dq}$

Example

Suppose the demand curve for a product is given by $q = 1350 - 5p$ and the cost function is $C(q) = 60q + 4q^2$. Find the quantity q that maximize profit.

1. $MR = \frac{dR}{dq}$

$R(q) = pq$ and need to substitute p for q.

Since $q = 1350 - 5p \Rightarrow 5p = 1350 - q \Rightarrow p = 270 - \frac{1}{5}q$

So, $R(q) = (270 - \frac{1}{5}q) \cdot q = \underline{270q - \frac{1}{5}q^2}$

& $\frac{dR}{dq} = 270 - \frac{2}{5}q$

2. Find Marginal cost:

$$MC = \frac{dC}{dq} = \frac{d}{dq}(60q + 4q^2) = 60 + 8q$$

3. Set $MC = MR$.

$$\Rightarrow \frac{dC}{dq} = \frac{dR}{dq} \Rightarrow 60 + 8q = 270 - \frac{2}{5}q$$

$$\Rightarrow 8q + \frac{2}{5}q = 270 - 60$$

$$P = R - \tilde{C}$$
$$= R(q) - C(q)$$

$$\Rightarrow \frac{42}{5}q = 210$$

$$\Rightarrow q = 25$$

So, profit is maximized when 25 goods are sold.