Last time we looked at Chain rule

Eg: $\frac{d}{dx}\ln(\sin(x))$

$f(u) = \ln(u)$

$u(x) = \sin(x)$

$$\boxed{\frac{d}{dx} f(u(x)) = f'(u(x)) \cdot u'(x)}$$

$$\frac{d}{dx} e^x = e^x$$

Logarithmic differentiation

Derivative of $\log_e(x)$ $(\ln(x))$

We want to compute $\frac{d}{dx}\ln(x)$. Let $y = \ln(x)$

function of x.

e on both sides

Take exponential on both sides:

$$e^y = e^{\log_e(x)}$$

$$\Rightarrow e^{y(x)} = x$$

Continued

Now we take $\frac{d}{dx}$ both sides:

$$\underbrace{\frac{d}{dx} e^{y(x)}} = \frac{d}{dx} x = 1$$

$$\hookrightarrow \frac{d}{dx} f(y(x)) \qquad \left(\underbrace{f(y) = e^{y}} \right)$$

So, we can use chain rule:

$$\Rightarrow \frac{d}{dy} f(y) \cdot y'(x) = 1$$

$$\Rightarrow e^{y} \cdot y'(x) = 1 \Rightarrow y'(x) = \frac{1}{e^{y}} = \frac{1}{x}$$

As expected!

Q: What is $\frac{d}{dx}\log_a(x)$, $a > 1$?

e^x
a^x?
$= y$

Write $\log_a(x)$ in terms of $\ln(x)$

Let $y = \log_a(x)$.

(a^x and $\log_a x$ are inverse functions)

$\Rightarrow \quad a^y = a^{\log_a(x)}$

$\Rightarrow \quad a^y = x$

$\Rightarrow \quad \ln(a^y) = \ln(x)$

$\Rightarrow \quad y\ln(a) = \ln(x) \Rightarrow y = \frac{1}{\ln(a)} \cdot \ln(x).$

So, $y'(x) = \frac{d}{dx}\log_a(x) = \frac{1}{\ln(a)} \cdot \frac{d}{dx}\ln(x) = \frac{1}{x\ln(a)}$

Q: What is $\frac{d}{dx} a^x$, where $a > 0$ is a constant?

Let $y = a^x$

Take log of both sides.

$$\Rightarrow \log_e(y) = \log_e(a^x) = x \underbrace{\log_e(a)}_{\text{constant}}$$

one approach: take $\frac{d}{dx}$ on both sides

onther approach: take exponent on both sides

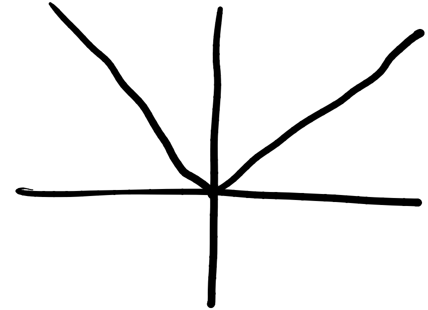
$$y = e^{x \cdot \log_e(a)}$$

$$\frac{dy}{dx} = e^{x \cdot \log_e(a)} \cdot \frac{d}{dx}\left(x \cdot \log_e(a)\right) = \log_e(a) \cdot e^{x \ln(a)}$$

$\frac{d}{dx} \ln|x| = ?$

$$\frac{d}{dx} e^{ax} = a e^{ax} \quad \text{(prove this!)}$$

Q: Compute $\frac{d}{dx} \ln|x|$.



Case 1: Assume $x > 0$:

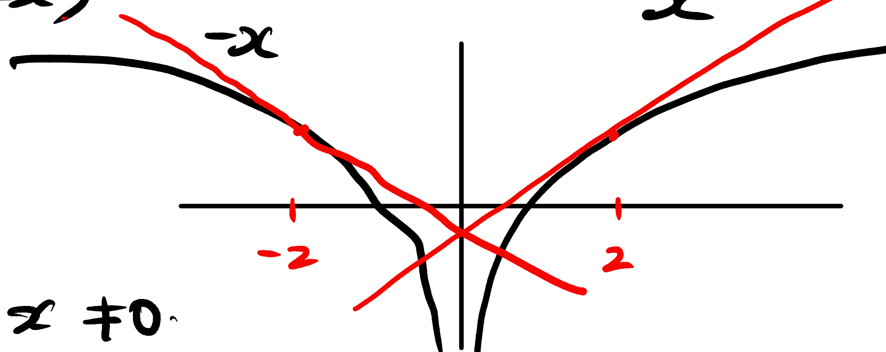
$$\frac{d}{dx} \ln|x| = \frac{d}{dx} \ln(x) = \frac{1}{x}$$

for $x<0$, $|x| = -x$

for $x>0$, $|x| = x$

Case 2: Assume $x < 0$:

$$\frac{d}{dx} \ln|x| = \frac{d}{dx} \ln(-x) = \frac{1}{-x} \cdot (-1) = \frac{1}{x}$$



So, $\frac{d}{dx} \ln|x| = \frac{1}{x}$

for $x \neq 0$.

Logarithmic differentiation

Differentiation technique that involves taking $\log_e$ of both sides and differentiating.

- Used when the function is a product of functions

eg: $y = \sqrt{x}\, e^{x^2}(x^2-1)^{9}$ 8

We could use product rule to differentiate this.

So, take absolute value first:

$$|y| = |\sqrt{x}|\ |e^{x^2}|\ |(x^2-1)^9|$$

$$\Rightarrow \ln|y| = \ln\left(|\sqrt{x}|\ |e^{x^2}|\ |(x^2-1)^9|\right)$$

$$= \ln|x^{\frac{1}{2}}| + \ln e^{x^2} + \ln|x^2-1|^9$$

$$= \frac{1}{2}\ln|x| + x^2 + 9\ln|x^2-1|.$$

Take $\frac{d}{dx}$ both sides:

$$\ln|y| = \frac{1}{2}\ln|x| + x^2 + 9\ln|x^2-1|$$

$$\Rightarrow \quad \frac{1}{y}\cdot y'(x) = \frac{1}{2x} + 2x + \frac{9}{|x^2-1|}\cdot 2x$$

chain rule

$$\Rightarrow \quad \frac{dy}{dx} = \left(\sqrt{x}\, e^{x^2}(x^2-1)^9\right)\cdot\left(\frac{1}{2x} + 2x + \frac{18x}{|x^2-1|}\right)$$

Another use of logarithmic differentiation is when

$$\frac{d}{dx}\left(\text{variable}^{\text{variable}}\right) \qquad \ln(x^x)$$

Find $\frac{d}{dx}\left(\ln(x)\right)^x \neq x \ln(x)^{x-1}$

power rule does not apply (ax^r)

Correct way: take logarithm on both sides:

$$y = (\ln(x))^x$$

$$\Rightarrow \ln(y) = x \cdot \ln(\ln(x)) \qquad \left(\frac{f(y) = \ln(y)}{y(x)}\right)$$

Take derivative wrt x on both sides:

$$\Rightarrow f'(y) \cdot y'(x) = \frac{d}{dx}\left(x \ln(\ln(x))\right)$$

$$\Rightarrow \frac{1}{y} \cdot y'(x) = 1 \cdot \ln(\ln(x)) + x \frac{d}{dx}\left(\ln(\ln(x))\right)$$

$$\frac{1}{y}\cdot y'(x) = 1\cdot \ln(\ln(x)) + x\,\frac{d}{dx}\big(\ln(\ln(x))\big)$$

$$\left.\begin{aligned} f(u) &= \boxed{\ln(u)} \\ u(x) &= \underline{\ln(x)} \\ &f(u(x)) \end{aligned}\right.$$

$$\frac{1}{y}\cdot y'(x) = \ln(\ln(x)) + x\cdot f'(u(x))\cdot u'(x)$$

$$\quad'' \qquad\qquad '' \qquad + \cancel{x}\cdot \frac{1}{u(x)}\cdot\frac{1}{\cancel{x}}$$

$$\frac{1}{y}\cdot y'(x) = \ln(\ln(x)) + \frac{1}{\ln(x)}$$

$$y'(x) = \Big(\underset{(\ln(x))^x}{y}\Big)\cdot\left(\ln(\ln(x)) + \frac{1}{\ln(x)}\right)$$