Implicit differentiation

Eg: Find the equation of the tangent line to

$$y = y^3 + xy + x^3 \quad \text{at } x = 1,\ y = ?$$

Note that we cannot directly solve for y interms of x here.
Use implicit differentiation here, i.e. differentiate both sides with respect to x and use chain rule.

for $\frac{d}{x} y^3$

outer: $f(y) = y^3$

inner: $y(x)$

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(y^3 + xy + x^3\right) \rightarrow$$

Key step $\longrightarrow$ $y' = \frac{d}{dy} y^3 \cdot \frac{dy}{dx} + \left(1 \cdot y + x \frac{dy}{dx}\right) + 3x^2$

$$= 3y^2 y' + y + xy' + 3x^2$$

$y' = (y + 3x^2) / (1 - 3y^2 - x)$

or $\quad y' = 3y^2 y' + y + xy' + 3x^2 \leftarrow$ solve for y'

Example contd.

$$y' = 3y^2y' + y + xy' + 3x^2$$
$$\Rightarrow y'(1-3y^2-x) = y+3x^2$$
$$\Rightarrow y' = (y+3x^2)/(1-3y^2-x)$$

To find y at $x=1$:

So, $y = y^3 + xy + x^3$

$$\Rightarrow \cancel{y} = y^3 + \cancel{y} + 1$$

$$\Rightarrow y^3 = -1 \Rightarrow y = -1$$

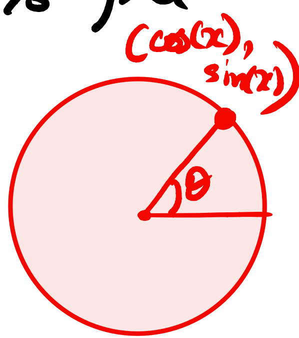
So, we are find tangent line at $(1,-1)$.

Thus, $y' = (-1 + 3\cdot 1^2)/\left(1-3(-1)^2-(1)\right) = -2/3$.

So, to find the tangent line: $y = -\frac{2}{3}x + b$. To find b, plug in $(1,-1)$:

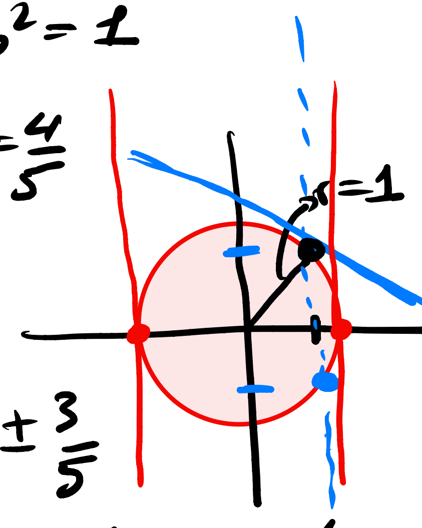
$$\Rightarrow -1 = -\frac{2}{3}(1) + b \Rightarrow b = -1 + \frac{2}{3} = -\frac{1}{3}$$

Thus, the tangent line is: $y = -\frac{2}{3}x - \frac{1}{3}$



Tangent line to a circle

Let (x_0, y_0) be a point on the circle $x^2+y^2=1$



Ex) Find the equation of tangent line at $x=\frac{4}{5}$

and $y \geq 0$

- First find $y \geq 0$ and $x = 4/5$

$$y^2 + \left(\frac{4}{5}\right)^2 = 1 \Rightarrow y^2 + \frac{16}{25} = 1 \Rightarrow y = \pm\frac{3}{5}$$

- Find the slope at $(4/5, 3/5)$: Implicit differentiation

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(1)$$

$$\Rightarrow 2x + 2y\,y' = 0 \Rightarrow y' = -x/y$$ only valid for $y \neq 0$

Tangent line to circle

so, $y' = -x/y$ and at $\left(\frac{4}{5}, \frac{3}{5}\right)$, $y' = -\frac{4/5}{3/5} = -\frac{4}{3}$

• Last finding equation of tangent line:

$$y = -\frac{4}{3}x + b$$

plugging in $\left(\frac{4}{5}, \frac{3}{5}\right)$ to find b:

$$\frac{3}{5} = -\frac{4}{3} \cdot \frac{4}{5} + b \Rightarrow b = \frac{3}{5} + \frac{16}{25} = \frac{25}{15} = \frac{5}{3}$$

so, tangent line at $\left(\frac{4}{5}, \frac{3}{5}\right)$ to the circle $x^2+y^2=1$:

$$\boxed{y = -\frac{4}{3}x + \frac{5}{3}}$$

Tangent line to circle at an arbitrary point (x_0, y_0):

(b) Find the tangent line at (x_0, y_0) to $x^2+y^2=1$:

$$y' = -x/y \Rightarrow y' = -x_0/y_0 \text{ for } y_0 \neq 0$$

and the equation of tangent line at (x_0, y_0) satisfies:

$$y = -\frac{x_0}{y_0} x + b$$

$$\Rightarrow y_0 = -\frac{x_0}{y_0} x_0 + b \Rightarrow b = y_0 + \frac{x_0^2}{y_0} \Rightarrow b = \frac{y_0^2 + x_0^2}{y_0}$$

$$= 1/y_0$$

So, equation of tangent line is:

$$y = -\frac{x_0}{y_0} \cdot x + \frac{1}{y_0} \text{ for } y_0 \neq 0, \quad x = \pm 1 \text{ at } y = 0$$

Contd:

$x^2+y^2=1$

Note that we can write down:

$$y = -\frac{x_0}{y_0}x + \frac{1}{y_0} \quad \infty$$

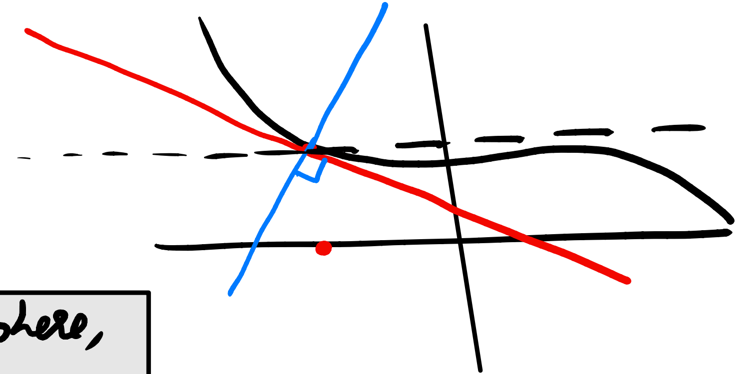
$$y y_0 + x x_0 = 1$$

If $y_0 = 0$, we get $x x_0 = 1 \Rightarrow x = \pm 1$

So, the general implicit equation of tangent line is:

$$y y_0 + x x_0 = 1$$ at (x_0, y_0) to $x^2+y^2=1$.

Equation of normal line



The normal line is perpendicular to tangent line.

$$\boxed{\text{Slope of normal line} = -\frac{1}{m} \text{ where, } m \text{ is the slope of the tangent line}}$$

Eg: Find the equation of the normal line to the curve

$$y^2 e^{2x} = 3y + x^2 \text{ at } (0,3)$$

- Find the slope of tangent line $\rightarrow$ Implicit diff.
- Find the slope of normal line $\rightarrow -\frac{1}{m}$
- Find the equation of the line $\longrightarrow y = -\frac{1}{m}x + b$

Example contd.

- Slope of tangent line:

$$\frac{d}{dx}\left(y^2 e^{2x}\right) = \frac{d}{dx}\left(3y + x^2\right)$$

$$\Rightarrow 2y\,y'\,e^{2x} + y^2\,2e^{2x} = 3y' + 2x$$

$$\Rightarrow y'(2y\,e^{2x} - 3) = 2x - y^2\,2e^{2x}$$

$$\Rightarrow y' = (2x - 2y^2 e^{2x}) / (2y\,e^{2x} - 3)$$

$\boxed{(0,3)}$

so, at $(0,3)$, $y' = -18/3 = -6$

- Slope of normal line is $1/6$
- equation of normal line: $y = \frac{1}{6}x + b$ plug in $(0,3)$

$$\Rightarrow 3 = b$$

$$\Rightarrow y = \frac{1}{6}x + 3$$

Example

Find y' by implicitly differentiating:

$$\tan(xy^2) = 3x + y^2$$

$$\Rightarrow \frac{d}{dx}\left(\tan(xy^2)\right) = \frac{d}{dx}\left(3x + y^2\right)$$

$$\Rightarrow \sec^2(xy^2) \cdot \frac{d}{dx}(xy^2) = 3 + 2yy'$$

$$\Rightarrow \sec^2(xy^2) \cdot \left(y^2 + x2yy'\right) = 3 + 2yy'$$

$$\Rightarrow \sec^2(xy^2)\,y^2 + 2xy\sec^2(xy^2)\,y' = 3 + 2yy'$$

$$\Rightarrow y' = \frac{\left(3 - \sec^2(xy^2)\,y^2\right)}{\left(2xy\sec^2(xy^2) - 2y\right)}$$